# Thermodynamical Approach for Small-Scale Parametrization in 2D Turbulence

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We propose a model of turbulent viscosity which preserves all the known conservation laws of the two-dimensional incompressible Euler equation, and is invariant by changes of reference frames. This model is derived by a systematic procedure, using a principle of maximum entropy production. [S0031-9007(97)02915-3]

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A fundamental difficulty in fluid turbulence is the development of motion at very small scales, down to the viscous dissipation scale, while computations can only provide locally averaged field, at the scale of the numerical mesh. The interaction of these explicit scales with the subgrid scales must be modeled in a statistical sense. Various forms of *turbulent viscosity* have been empirically introduced for this purpose, modeling the energy transfers toward the subgrid scales. In two-dimensional (2D) turbulence, such energy cascade is forbidden by the conservation of vorticity of fluid particles. However, small scale fluctuations are still produced in the vorticity field, and specific models have been proposed [1], but they do not preserve all the conservation laws for 2D fluid motion.

The goal of this paper is to derive, by a systematic procedure, an evolution equation for the explicit scales, smoothing out the subgrid scales, while *preserving all the known conservation laws of the 2D Euler equations.* The guiding idea is that turbulent diffusion is an irreversible process, producing disorder or entropy. Developments of a statistical equilibrium theory [2,3] for 2D perfect fluids support this view, and provide an explicit expression for the entropy. It has been proposed [4–6] that subgrid fluctuations *drive the system towards this statistical equilibrium.* The resulting relaxation equations preserve the conservation laws of the Euler equations (like energy), but only as *global* constraints. Furthermore, these equations are not invariant by all changes of reference frame (rotating or translating), which is not satisfactory from a fundamental point of view, and may lead to practical flaws in large systems. The purpose of the present paper is to derive more general relaxation equations, preserving all the invariance properties and conservation laws of the Euler equations in a *local* sense.

We start with the Euler equations describing 2D inviscid and incompressible flows

$$\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \mathbf{u}) = 0, \qquad \mathbf{u} = -\mathbf{z} \wedge \nabla \psi,$$
$$\omega = -\Delta \psi, \tag{1}$$

where $\psi$ is the stream function and $\omega \mathbf{z} = \nabla \wedge \mathbf{u}$ the vorticity. This equation is known to have solutions for all times for any regular initial condition (but rapidly develops finer and finer vorticity filaments). Therefore it is not necessary to introduce a physical viscosity (as long as the viscous dissipation scale is much below the cutoff for the explicit scales). The vorticity of each fluid particle is conserved, implying the conservation of the *global* probability distribution of vorticity $\gamma(\sigma)$ (i.e., the total area fraction occupied by each vorticity level $\sigma$). The other conserved quantities are the energy $E = \frac{1}{2} \int \omega\, \psi d^2\mathbf{r}$, and, in the infinite domain, the angular momentum $L = \int \omega r^2 d^2\mathbf{r}$ and the impulse $\mathbf{P} = \int \mathbf{r} \wedge \omega \hat{\mathbf{z}} d^2\mathbf{r}$.

We are interested in the smoothed vorticity field $\overline{\omega}$, obtained as a local average, over a cutoff scale $d$, of the vorticity $\omega$. We also define the corresponding stream function $\overline{\psi}$, solution of $-\Delta\overline{\psi} = \overline{\omega}$, and $\overline{\mathbf{u}} = -\mathbf{z} \wedge \nabla\overline{\psi}$. Applying local averaging to the Euler equation (1) leads to an equation of vorticity transport

$$\frac{\partial \overline{\omega}}{\partial t} + \nabla \cdot (\overline{\omega}\, \overline{\mathbf{u}}) = -\nabla \cdot \mathbf{J}_\omega, \tag{2}$$

where the flux $\mathbf{J}_\omega$ depends on the local fluctuations $\tilde{\mathbf{u}}$ and $\tilde{\omega}$ ($\mathbf{J}_\omega = \overline{\tilde{\mathbf{u}}\tilde{\omega}}$), and must be obtained by a closure model. The conservation laws for energy, angular momentum, and impulse, respectively, lead to the constraints

$$\dot{E} = \int \mathbf{J}_\omega \cdot \nabla\overline{\psi}\, d^2\mathbf{r} = 0, \tag{3}$$

$$\dot{L} = \int 2\mathbf{J}_\omega \cdot \mathbf{r}\, d^2\mathbf{r} = 0, \qquad \dot{\mathbf{P}} = \int \mathbf{J}_\omega \wedge \mathbf{z}\, d^2\mathbf{r} = 0, \tag{4}$$

where we have neglected in (3) the energy $\epsilon = (1/2)\overline{\tilde{\psi}\tilde{\omega}}$ of the local fluctuations. We can indeed estimate that $\tilde{\psi} \sim \tilde{\omega} d^2$, so that $\epsilon \sim \tilde{\omega}^2 d^2$, and $\epsilon/E \sim (d/L)^2 \ll 1$, where $L$ is a scale of motion.

In order to close (2), we need a statistical description of the subgrid-scale vorticity fluctuations. For that purpose, we introduce a (time dependent) *local* probability density (i.e., area proportion) $\rho(\mathbf{r}, \sigma)$ of finding the vorticity level $\sigma$ in a small neighborhood of the position $\mathbf{r}$. This density satisfies the local normalization condition $\int \rho(\mathbf{r}, \sigma) d\sigma = 1$; its first moment is the locally averaged vorticity, and its second moment determines the mean square $\omega_2$ of the vorticity fluctuations

$$\overline{\omega}(\mathbf{r}) = \int \rho(\mathbf{r},\sigma)\sigma\, d\sigma\,,$$
$$\omega_2(\mathbf{r}) = \int \rho(\mathbf{r},\sigma)\,(\sigma - \overline{\omega})^2\, d\sigma\,. \tag{5}$$

The vorticity level $\sigma$ of each fluid parcel is conserved, like a "chemical" with concentration $\rho$, and satisfies a transport equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overline{\mathbf{u}}) = -\nabla \cdot \mathbf{J}\,, \tag{6}$$

with $\mathbf{J} \cdot \mathbf{n} = 0$ at the fluid boundaries. This is an exact local expression for the conservation of the global probability $\gamma(\sigma) = \int \rho(\mathbf{r},\sigma)\, d^2\mathbf{r}$. The local normalization condition and the conservation law (2) for $\overline{\omega}$ imply

$$\int \mathbf{J}(\mathbf{r},\sigma)\, d\sigma = 0\,, \tag{7}$$

$$\mathbf{J}_\omega = \int \mathbf{J}(\mathbf{r},\sigma)\sigma\, d\sigma\,. \tag{8}$$

Using these expressions, the form (6) with constraints (3) and (4) preserves all the conservation laws of the Euler equations.

In order to determine the diffusion currents $\mathbf{J}$, we now define the entropy [2]

$$S = -\int \rho(\mathbf{r},\sigma)\ln \rho(\mathbf{r},\sigma)\, d^2\mathbf{r}\, d\sigma\,, \tag{9}$$

which characterizes the "number" of possible vorticity fields (microscopic states) leading to a given density field $\rho(\mathbf{r},\sigma)$ (macroscopic state). After sufficient time the system has an overwhelming probability to reach an equilibrium state, maximizing the entropy, with the constraints imposed by the conserved quantities. During the process of relaxation toward equilibrium, we expect that the entropy always increases

$$\dot{S} = -\int \mathbf{J} \cdot \nabla(\ln \rho)\, d^2\mathbf{r}\, d\sigma > 0\,. \tag{10}$$

Furthermore, this increase is likely to be maximum with appropriate constraints. These are the conservation laws, and also kinematic constraints: the diffusion fluxes $\mathbf{J}$ cannot take arbitrary large values as they result from the transport by the subgrid-scale velocity fluctuations. This is the maximum entropy production principle (MEPP) [4,5]: for a given density field $\rho(\mathbf{r},\sigma)$, the system distributes its currents $\mathbf{J}$ in order to maximize its rate of entropy production $\dot{S}$ while satisfying the constraints (3) and (7) and the inequality $\int (J^2/2\rho)\, d\sigma \le C(\mathbf{r})$. This variational problem yields optimal currents of the form [4]

$$\mathbf{J} = -D(\mathbf{r})[\nabla\rho + \beta(t)\,(\sigma - \overline{\omega})\rho\nabla\overline{\psi}\,]\,, \tag{11}$$

where $\beta(t)$ is the (global) Lagrange multiplier corresponding to energy conservation, obtained by introducing the condition (3) in (11), which yields [4]

$$\beta(t) = -\int D\nabla\overline{\psi} \cdot \nabla\overline{\omega}\, d^2\mathbf{r} \Big/ \int D\omega_2(\nabla\overline{\psi})^2\, d^2\mathbf{r}\,. \tag{12}$$

The evolution equation (6) is then fully determined; it has the form [7] of a Fokker-Planck equation, with an ordinary diffusive term [the first term in the bracket of (11)] and a systematic drift due to a "potential" gradient $\nabla\overline{\psi}$. The additional constraints (4) must be introduced when the boundary is invariant by a rotation or translation, and $\overline{\psi}$ then becomes a stream function in a rotating or translating frame of reference.

The MEPP is able to give the general form of the relaxation equations (it can be viewed as a variational formulation of ordinary linear thermodynamics) but cannot predict by itself the expression of the diffusion coefficient $D(\mathbf{r})$ [related to the unknown bound $C(\mathbf{r})$ for the diffusion currents]. However, diffusion is due to local velocity fluctuations $\tilde{u} \sim \omega_2^{1/2} d$ over scale $d$, so dimensional arguments lead to an expression

$$D(\mathbf{r},t) = Kd^2\omega_2^{1/2}, \tag{13}$$

with a nondimensional constant $K$ of order unity, which can be specified by a simple stochastic model [5,7].

This system evolves toward equilibrium states, for which the diffusion currents (11) vanish [4]. Then, if $D \neq 0$, $(\nabla\rho)/\rho = -\beta(\sigma - \overline{\omega})\nabla\overline{\psi}$, so that for two vorticity levels $\sigma$ and $\sigma'$, $\nabla \ln[\rho(\mathbf{r},\sigma)/\rho(\mathbf{r},\sigma')] = \nabla[-\beta(\sigma - \sigma')\overline{\psi}]$. Introducing a constant of integration $\alpha(\sigma)$, and using the local normalization condition, we get $\rho(\mathbf{r},\sigma) = \exp[-\alpha(\sigma) - \beta\sigma\overline{\psi}(\mathbf{r})]/\int \exp[-\alpha(\sigma') - \beta\sigma'\overline{\psi}(\mathbf{r})]\, d\sigma'$. This is the expression for equilibrium states, obtained as well by maximizing entropy with the constraints of the conservation laws [2]. $\beta$ is then interpreted as the inverse of a temperature. Notice, however, that the resulting equilibrium state is possibly restricted to a subregion of space, surrounded by irrotational fluid (where $\omega_2 = 0$ so that $D = 0$). It may therefore differ from the global equilibrium in the whole domain, and represents organization into isolated vorticity structures [5,8].

In summary, the previous equations smooth the Euler equation, while preserving the constraints imposed by the dynamics. In particular, the energy and total circulation $\int \overline{\omega}\, d^2\mathbf{r}$ are exactly conserved, and starting from a smooth initial condition $\omega_0(\mathbf{r})$, the enstrophy must decrease (i.e., $\int \overline{\omega}^2\, d^2\mathbf{r} \le \int {\omega_0}^2\, d^2\mathbf{r}$), and $\max(\overline{\omega}) \le \max(\omega_0)$, $\min(\overline{\omega}) \ge \min(\omega_0)$. The entropy increases (with an optimal rate), leading to a (possibly restricted) equilibrium state. This has been obtained at the price of a new variable, the vorticity level $\sigma$, but in the case of discrete vorticity levels, $a_i$, $i = 1, N$, the system simplifies into a set of $N$ equations. As an alternative approach, we can take the first moments (in $\sigma$) of (6) and (11), closing the resulting hierarchy by a maximum entropy hypothesis [5]. These relaxation equations, derived from the MEPP, compete very well with direct Navier-Stokes simulations [5] but, of course, demand less resolution since the small scales have been modeled in an optimal (statistical) way.

However, these relaxation equations have a conceptual drawback, as they do not respect the Galilean invariance: the term $\nabla\overline{\psi}$ is modified by a change of reference frame. Furthermore, energy is only conserved by the global integral relation (12), while for well separated active subparts, we expect energy to be conserved in each subpart (with its own temperature $\beta$), rather than globally.

The aim of the present paper is to cure these problems by reformulating the MEPP in a fully local form. We introduce currents of energy $\mathbf{J}_\epsilon$, angular momentum $\mathbf{J}_\lambda$, and impulse $\Pi_{ij}$ (in the $j$ direction) and rewrite the constraints (3) and (4) as

$$\mathbf{J}_\omega \cdot \nabla\overline{\psi} = \nabla \cdot \mathbf{J}_\epsilon \,, \tag{14}$$

$$\mathbf{J}_\omega \cdot \mathbf{r} = \nabla \cdot \mathbf{J}_\lambda \,, \tag{15}$$

$$J_{\omega i} = \partial_j \Pi_{ij} \,, \tag{16}$$

with the conditions that the normal components of these currents vanish at the boundaries. At this stage this is equivalent to the global constraints (3) and (4), but the novelty will arise by bounding these currents.

However, there is an immediate difficulty, as these new fluxes are clearly not invariant by a change of reference frame. In fact, there is no reason to bound $\mathbf{J}_\epsilon$, $\mathbf{J}_\lambda$, and $\Pi_{ij}$ rather than any linear combination of them. The choice will be dictated by considerations of symmetry, as we want our relaxation equations to satisfy the invariance properties of the Euler equations, that is, (i) the invariance by translation of the coordinates, (ii) the invariance by rotation of the coordinates, (iii) the invariance by gauge transformation $\psi \to \psi + C$, (iv) the Galilean invariance $\psi \to \psi + U_i x_i$, and (v) the invariance by rotation of the referential $\psi \to \psi + \Omega_* r^2$. (This is specific to 2D incompressible flows, for which the Coriolis and centrifugal forces are exactly balanced by pressure forces.)

Relation (16) satisfies these properties, while (15) depends on the coordinate origin, but the current $J'_{\lambda j} = J_{\lambda j} - x_i \Pi_{ij}$ satisfies the invariant relation

$$\Pi_{ii} = -\nabla \cdot \mathbf{J}'_\lambda \tag{17}$$

(with summation over repeated indices), obtained by combining (15) and (16). Relation (14) is clearly not invariant by changes of reference frame, but this property is recovered for the new current $J'_{\epsilon j} = J_{\epsilon j} - \Pi_{ij}\partial_i\overline{\psi} - \frac{1}{2}(\Delta\overline{\psi})J'_{\lambda j}$, which satisfies

$$\Lambda_{ij}\Pi_{ij} - \tfrac{1}{2}\mathbf{J}'_\lambda \cdot \nabla(\Delta\overline{\psi}) = \nabla \cdot \mathbf{J}'_\epsilon \,, \tag{18}$$

obtained by combining (14), (16), and (17), and denoting

$$\Lambda_{ij} \equiv -\partial^2_{ij}\overline{\psi} + \tfrac{1}{2}\Delta\overline{\psi}\delta_{ij} \,. \tag{19}$$

Under the form (16), (17), and (18) the constraints on energy, angular momentum, and impulse now satisfy all the required invariance properties. Let us furthermore introduce the decomposition $\Pi_{ij} = S_{ij} + A_{ij} + \pi\delta_{ij}$ involving a symmetric traceless tensor ($S_{ij} = S_{ji}$ and $S_{ii} = 0$), and an antisymmetric tensor ($A_{ij} = -A_{ji}$), so that the local constraints (16), (17), and (18) become

$$\Lambda_{ij}S_{ij} + \tfrac{1}{2}\mathbf{J}'_\lambda \cdot \nabla\overline{\omega} = \nabla \cdot \mathbf{J}'_\epsilon \,, \tag{20}$$

$$\pi = -\tfrac{1}{2}\nabla \cdot \mathbf{J}'_\lambda \,, \tag{21}$$

$$J_{\omega i} = \partial_j(S_{ij} + A_{ij} + \pi\delta_{ij}) \,. \tag{22}$$

The optimal currents are then determined by maximizing the rate of entropy production (10) under these constraints (20)–(22), the normalization condition (7), and the inequalities $\int (J^2/2\rho)d\sigma \le C(\mathbf{r})$, $J^2_{\epsilon,\lambda} \le C_{\epsilon,\lambda}(\mathbf{r})$, $S_{ij}S_{ij} \le C_S(\mathbf{r})$, $A_{ij}A_{ij} \le C_A(\mathbf{r})$, and $\frac{\pi^2}{2} \le C_\pi(\mathbf{r})$ preventing the diffusion currents from taking arbitrarily large values. This variational problem is treated by introducing Lagrange multipliers, $\beta$, $\gamma$, $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, $1/D$, $1/\chi_\epsilon$, $1/\chi_\lambda$, $1/\chi_S$, $1/\chi_A$, $1/\chi_\pi$ for each constraint. The resulting optimal currents are

$$\mathbf{J} = -D[\nabla\rho + \rho(\sigma - \overline{\omega})\boldsymbol{\eta}] \,, \tag{23}$$

$$\mathbf{J}'_\epsilon = -\chi_\epsilon\nabla\beta \,, \tag{24}$$

$$\mathbf{J}'_\lambda = \chi_\lambda\left[\nabla\gamma - \frac{\beta}{2}\nabla\overline{\omega}\right], \tag{25}$$

$$\pi = -\chi_\pi(2\gamma + \nabla \cdot \boldsymbol{\eta}) \,, \tag{26}$$

$$S_{ij} = \chi_S\left[-\beta\Lambda_{ij} - \frac{1}{2}\left(\frac{\partial\eta_i}{\partial x_j} + \frac{\partial\eta_j}{\partial x_i}\right) + \frac{1}{2}\nabla \cdot \boldsymbol{\eta}\delta_{ij}\right], \tag{27}$$

$$A_{ij} = -\frac{\chi_A}{2}\left(\frac{\partial\eta_i}{\partial x_j} - \frac{\partial\eta_j}{\partial x_i}\right). \tag{28}$$

These relations express that the fluxes must be linear functions of the density $\rho$ and the "thermodynamic potentials" $\beta$, $\gamma$, and $\eta_i$. The latters appeared as Lagrange multipliers in the MEPP, and do not correspond to any local equilibrium, unlike in usual thermodynamics (in particular, we have no local internal energy).

Substituting these expressions (23)–(28) into (20)–(22), we obtain a system of four equations determining the four potentials $\eta_i$, $\gamma$, and $\beta$. With the particular (and convenient) choice $\chi_A = \chi_S$ and $\chi_\pi = \chi_S/2$, this system simplifies into

$$\partial_j(\chi_S\partial_j\eta_i) - D\omega_2\eta_i + \partial_j(\chi_S\Lambda_{ij}\beta) + \partial_i(\chi_S\gamma) = D\partial_i\overline{\omega} \,,$$
$$\nabla(\chi_\epsilon\nabla\beta) - \left[\chi_S\Lambda_{ij}\Lambda_{ij} + \frac{\chi_\lambda}{4}(\nabla\overline{\omega})^2\right]\beta - \frac{\chi_S}{2}\Lambda_{ij}(\partial_j\eta_i + \partial_i\eta_j) + \frac{\chi_\lambda}{2}\nabla\overline{\omega} \cdot \nabla\gamma = 0 \,,$$
$$\nabla(\chi_\lambda\nabla\gamma) - 2\chi_S\gamma - \chi_S\nabla \cdot \boldsymbol{\eta} - \nabla\left(\frac{\chi_\lambda}{2}\beta\nabla\overline{\omega}\right) = 0 \,. \tag{29}$$

For given diffusion coefficients, these are linear elliptical equations for each of the four unknown (with additional crossed terms involving first order derivatives). The boundary conditions correspond to vanishing normal currents, related to the unknown by (24)–(28). Defining the coordinates $(\chi, \zeta)$ tangential and normal to the wall, this yields

$$\frac{\partial \beta}{\partial \zeta} = 0, \qquad \frac{\partial \gamma}{\partial \zeta} = \frac{\beta}{2}\frac{\partial \overline{\omega}}{\partial \zeta},$$
$$\frac{\partial \eta_\chi}{\partial \zeta} = \beta \frac{\partial \overline{u}_\zeta}{\partial \zeta}, \qquad \frac{\partial \eta_\zeta}{\partial \zeta} = -\beta \frac{\partial \overline{u}_\chi}{\partial \zeta} + \frac{\beta}{2}\overline{\omega} - \gamma, \tag{30}$$

which fully determines the system (29). The resulting vector $\boldsymbol{\eta}$ will specify the current $\mathbf{J}$ by (23), determining the time evolution by (6), together with the boundary condition $\mathbf{J} \cdot \mathbf{n} = 0$, i.e., $\partial \rho / \partial \zeta = -\rho(\sigma - \overline{\omega})\eta_\zeta$ (and the impermeability condition $\overline{\psi} = cte$).

The diffusivities are not given by the MEPP but must be positive in order to satisfy the increase of entropy. Indeed, the rate of entropy production (10) can be put under the form

$$\dot{S} = \int d^2\mathbf{r} \left[ \int \frac{\mathbf{J}^2}{D\rho} d\sigma + \frac{\mathbf{J}_\epsilon'^2}{\chi_\epsilon} + \frac{\mathbf{J}_\lambda^2}{\chi_\lambda} + \frac{S_{ij}^2}{\chi_S} + \frac{A_{ij}^2}{\chi_A} + \frac{\pi^2}{\chi_\pi} \right]. \tag{31}$$

The diffusivities must depend on local quantities, independent of the reference frame. The coefficient $D$ can still be estimated by (13), and it is reasonable to assume that all the other diffusivities are proportional to $D$, as they represent constraints on the vorticity diffusion. We shall also assume that each coefficient depends only on its conjugate Lagrange multiplier. Then dimensional analysis yields the following estimates

$$\chi_\epsilon \sim \frac{D}{\beta^2}, \qquad \chi_\lambda \sim \frac{D}{\gamma^2}, \qquad \chi_{S,A,\tilde{\pi}} \sim \frac{D}{\eta^2}, \tag{32}$$

providing a similar structure for the first term of each of the three equations (29). Of course this determination is partly arbitrary, but the behavior of the system is probably not very sensitive to these diffusivities, as checked in numerical computations [5,9].

Indeed the system tends to local equilibria, which are independent of the diffusivities, as discussed now. Suppose the system reaches a steady state in some region of space where $\omega_2 \neq 0$ (so that $D \neq 0$). Then we have $\dot{S} = 0$, and (31) implies that all the currents vanish. The condition (28) implies $\nabla \wedge \boldsymbol{\eta} = 0$, so that $\boldsymbol{\eta}$ must be the gradient of some function: $\boldsymbol{\eta} = -\nabla\phi$. From (24) and (25), $\beta$ must be uniform, and $\gamma - \frac{\beta}{2}\overline{\omega} = C$, where $C$ is a constant of integration, while (26) implies $\Delta\phi = 2\gamma$. Combining these relations, we find $\Delta(\phi + \beta\psi) = 2C$, so that $\phi + \beta\psi = \frac{C}{2}r^2 + \phi_0$, where $\phi_0$ satisfies the Laplace equation $\Delta\phi_0 = 0$. The condition $S_{ij} = 0$ implies $\partial_{ij}^2\phi_0 = 0$, and the only solution of the Laplace equation satisfying this condition is $\phi_0 = \mathbf{K}\mathbf{r}$. We can relabel the constants $C$ and $\mathbf{K}$ by $C = -\beta\Omega$ and $\mathbf{K} = \beta(\mathbf{z} \wedge \mathbf{V})$, so that the condition $\dot{S} = 0$ finally implies

$$\boldsymbol{\eta} = -\nabla\phi, \qquad \phi = -\beta\left(\overline{\psi} + \frac{\Omega}{2}r^2 - (\mathbf{V} \wedge \mathbf{r})_z\right). \tag{33}$$

Therefore the expression for $\mathbf{J}$ reduces to (11), in a reference frame rotating with angular velocity $\Omega$ and translating with velocity $\mathbf{V}$. As discussed before, the condition $\mathbf{J} = 0$ leads then to an equilibrium state in this moving frame of reference. In a small domain, the boundary conditions will enforce (33) even before equilibrium is reached, and we recover the relaxation equations of Refs. [4,5], involving the current (11) with the global constraint (12) (in a domain without special symmetry).

To summarize, our relaxation equations conserve all the constants of motion, respect all the invariance properties of the Euler equations, increase the entropy with an optimal rate until an equilibrium state is reached. These equations involve a diffusion flux for the local probability density $\rho$. This is a usual diffusion flux in $\nabla\rho$, corrected by a systematic drift, proportional to a vector $\boldsymbol{\eta}(\mathbf{r})$, determined as a solution of the elliptic system (29) with boundary conditions (30). In a strongly confined system, our model reduces to the previously obtained relaxation equations [4,5], but it should also predict organization restricted to several subregions of a large system, thanks to the local character of all the conservation laws. The same methods can be applied to quasigeostrophic systems [9] or Vlasov equations [7].

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