Jaynes-Cummings Model for a Trapped Ion in Any Position of a Standing Wave

Ying Wu and Xiaoxue Yang
Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China
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It is shown that the dynamics of a two-level ion in any position other than at the antinode of a standing wave that moves in a harmonic trapping potential can be described by Jaynes-Cummings model under the conditions of the rotating-wave approximation, the Lamb-Dicke limit, and the strong confinement limit. [S0031-9007(97)02997-9]

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Over the last two decades much attention has been focused on the Jaynes-Cummings (JC) model and its various nonlinear extensions in quantum optics. Such models are shown to exhibit interesting nonclassical effects, such as the collapse and revivals of Rabi oscillations [1–4], antibunched light [1,2], squeezing [1,2], inversionless light amplification [5], electromagnetic induced transparency [6], etc. These phenomena have wide applications in micromaser [7], microlaser [8], ultrahigh precision spectroscopy [1,9,10], etc. For instance, the collapse and revivals of Rabi oscillations may be utilized to realize quantum-nondemolition measurement of the photon statistics in a cavity [9]. The extensions of the JC model are mainly along two directions. On the one hand, one still considers a two-level atom and a single-mode quantized light field but taking multiphoton processes into account [1,11]. On the other hand, one can consider a three-level atom and multimode quantized cavity fields, and then turns this system into an effective two-level problem by the adiabatic elimination approximation [1,2] or perturbation transformation method [3]. It has recently been proved by one of us [4] that the system of a three-level atom interacting with two quantized fields in the Λ configuration can exactly be transformed into an effective two-level problem. Subsequently, we [11] have developed a unified and standardized method to solve analytically various linear and nonlinear JC models, and established the similarity between JC models and the one describing a spin-$1/2$ particle in an external magnetic field, which generalizes the well-known conclusion by Feynman *et al.* for the special case of linear JC model with the single-mode field treated classically to the one for the general situation of linear and nonlinear JC models with the quantized field(s).

Recently, Cirac *et al.* [12] have shown that the dynamics of a trapped and laser-cooled two-level ion, at the node of a standing wave, is described by the JC model, one of the paradigms of quantum optics mentioned above. Except for the rotating-wave approximation, the other conditions for such a description are that the vibrational amplitude of the ion motion is much less than the wavelength of the light (Lamb-Dicke limit), and that the trap frequency ν is much greater than the atomic decay rate Γ (strong confinement limit) which does not seem to have been achieved in current experiments. Given these conditions, they showed that the well-established field of ion trapping can be applied to the investigations of the JC model and of effects taken directly from cavity quantum electrodynamics (QED) studies. Moreover, there appear to be considerable advantages in the trapped-ion realization of the JC model over that with a cavity field mode, as the vibrational states (phonons) of the trap, replacing photons in the JC model of quantum optics, is only very weakly damped and, further, the strength of the coupling between the two-level ion and the phonons can be made very strong simply by increasing the intensity of the standing wave [12,13]. Based on this important relation between trapped ion dynamics and cavity QED, many fascinating proposals or ideas [12,14–10] have been put forward, for instance, the quantum-nondemolition measurement of the final temperatures or the vibrational energy of a trapped ion by applying a sequence of probe pulses [12] or a CW probe field [13], theoretical analysis [12,14,15] and experimental realization [10] of collapse and revivals, as well as nonclassical states of the ion motion, such as Fock, coherent, and squeezed states. However, there exists an important unanswered question whether the dynamics of a trapped ion in any other position, besides at the node, of a standing wave is still described by the JC model under the same conditions, which will be shown to be true in this paper.

In the situation where the trap frequency ν is much greater than the atomic decay rate (strong confinement limit), we can neglect the effect of the atomic decay rate and consider the dynamics of a trapped and laser-cooled two-level ion in a standing wave by Hamiltonian formalism. The Hamiltonian in this case reads ($\hbar = 1$) [12,14,15]

$$H = \nu a^\dagger a + \frac{\Delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x \cos(\eta x + \phi), \quad (1)$$

where ν is the trap frequency, $a, a^\dagger$ are destruction and creation operators for phonons or the vibrational states of the trap, $\Delta = \omega_0 - \omega_l$ is the detuning between the transition and laser frequencies, $\sigma_x = \sigma_{21} + \sigma_{12}$ and $\sigma_z = \sigma_{22} - \sigma_{11}$ are the two-level polarization and inversion

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operators, σ_{ii} are the level occupation operators and σ_{ij} $(i \neq j)$ are the transition operators from levels j to i satisfying $\sigma_{jk}\sigma_{mn} = \sigma_{jn}\delta_{mk}$ and $\sigma_{11} + \sigma_{22} = 1$, and $\sigma_{22} = (1 + \sigma_z)/2, \sigma_{11} = (1 - \sigma_z)/2$, Ω is the Rabi frequency proportional to the amplitude of the standing wave, η is Lamb-Dick parameter, $x = a + a^\dagger$ denotes a dimensionless position operator for the position of the ion, and ϕ accounts for the relative position of the ion in the standing wave. In particular, $\phi = \pi/2$ corresponds to an ion centered at a node of the standing wave. In the Lamb-Dicke limit, the Hamiltonian can be rewritten as $H = H_0 + H_{\text{int}}$ where

$$H_0 = \nu a^\dagger a + \frac{\Omega_\phi}{2}\sigma_x + \frac{\Delta}{2}\sigma_z\,,$$

$$H_{\text{int}} = -\eta\frac{\Omega\sin\phi}{2}(\sigma_{21} + \sigma_{12})x + O(\eta^2)\,, \quad (2)$$

where $\Omega_\phi = \Omega\cos\phi$, and $O(\eta^2)$ denotes the terms equal to and greater than the order of η^2, which will be neglected hereafter. Introducing the interaction picture by the relation $\tilde{Y} = \exp(iH_0t)Y\exp(-iH_0t)$, we, after some calculations, obtain $\tilde{x} = a\exp(-i\nu t) + a^\dagger\exp(i\nu t)$ and

$$\tilde{H}_0 = H_0 = \nu a^\dagger a + \frac{\Omega_t}{2}\Sigma_z\,, \quad (3)$$

$$\tilde{\sigma_x} = -\frac{\Delta}{2\Omega_t}[\Sigma_{21}\exp(i\Omega_t t) + \Sigma_{12}\exp(-i\Omega_t t)] + \frac{\Omega_\phi}{\Omega_t}\Sigma_z\,, \quad (4)$$

where

$$\Omega_t = \sqrt{\Omega_\phi^2 + \Delta^2}\,, \qquad \Sigma_z = \frac{\Omega_\phi}{\Omega_t}\sigma_x + \frac{\Delta}{\Omega_t}\sigma_z\,, \quad (5a)$$

$$\Sigma_{12} = \frac{1}{2}\left(\frac{\Delta}{\Omega_t}\sigma_x - \frac{\Omega_\phi}{\Omega_t}\sigma_z + \sigma_x\sigma_z\right), \quad (5b)$$

$$\Sigma_{21} = \frac{1}{2}\left(\frac{\Delta}{\Omega_t}\sigma_x - \frac{\Omega_\phi}{\Omega_t}\sigma_z - \sigma_x\sigma_z\right). \quad (5c)$$

Using the above results, we arrive at

$$\tilde{H}_{\text{int}} = -\eta\frac{\Omega\Delta\sin\phi}{2\Omega_t}(\{\Sigma_{21}a\exp[i(\Omega_t - \nu)t] + \Sigma_{12}a^\dagger\exp[-i(\Omega_t - \nu)t]\} + \{\Sigma_{21}a^\dagger\exp[i(\Omega_t + \nu)t] + \Sigma_{12}a\exp[-i(\Omega_t + \nu)t]\}) + \eta\frac{\Omega\Delta\cos\phi\sin\phi}{2\Omega_t}[a\exp(-i\nu t) + a^\dagger\exp(i\nu t)]\Sigma_z\,. \quad (6)$$

Under the rotating-wave approximation $\Omega_t, \nu \gg |\Omega_t - \nu|$, the last two rows in the above equation represent off-resonant terms and can be neglected compared with the resonant terms, i.e., the terms in the first row of the above equation. We therefore obtain the Hamiltonian in the interaction picture under the conditions of the rotating-wave approximation and Lamb-Dicke limit as well as strong confinement limit as follows:

$$\tilde{H} = \nu a^\dagger a + \frac{\Omega_t}{2}\Sigma_z - \eta\frac{\Omega\Delta\sin\phi}{2\Omega_t}(a\Sigma_{21} + a^\dagger\Sigma_{12})\,, \quad (7)$$

where the time dependence has been suppressed by going from the Shrödinger to the Heisenberg picture. This equation is the central result of this paper.

Let us now explain the physical meaning of the operators Σ_{ij} and illustrate that the above equation is the JC model for the interaction between phonons and the ion dressed by the standing wave. Using the definitions of Σ_{12}, Σ_{21}, and Σ_z and defining $\Sigma_{11} = \Sigma_{12}\Sigma_{21}$ and $\Sigma_{22} = \Sigma_{21}\Sigma_{12}$, one easily shows that these operators satisfy $\Sigma_{jk}\Sigma_{mn} = \Sigma_{jn}\delta_{mk}$, $\Sigma_{11} + \Sigma_{22} = 1$, $\Sigma_{22} = (1 + \Sigma_z)/2$, and $\Sigma_{11} = (1 - \Sigma_z)/2$. Their physical meanings are as follow: $\Sigma_z = \Sigma_{22} - \Sigma_{11}$ is the inversion operator for the two dressed levels of the ion (the two bare levels of the ion are dressed by the standing wave), Σ_{ii} are the dressed level occupation operators, and Σ_{ij} $(i \neq j)$ are the transition operators from dressed levels j to i. In other words, σ_{ij} correspond to the operators for the two bare levels of the ion while Σ_{ij} are those for its two dressed levels. Consequently, Eq. (7) indeed represents the JC model for the interaction between phonons and the ion dressed by the standing wave. In particular, in the circumstance of $\phi = \pi/2$ corresponding to the ion centered at a node of the standing wave, the operators Σ_{ij} for the dressed levels become σ_{ij} for the bare levels, and Eq. (7) reduces to the JC model derived previously by Cirac *et al.*

In summary, we have, in this paper, introduced the dressed-state description and shown that the dynamics of a two-level ion in a harmonic trapping potential oscillating not only around the node of a standing wave, as in the previous result [12], but also around any other position of the standing wave can be described by the Jaynes-Cummings model under the conditions of rotating-wave approximation, Lamb-Dicke limit, and strong confinement limit. This conclusion establishes a complete connection between the trapped-ion dynamics and the cavity QED within the framework of the JC model, and should, in our view, have important implications to both fields. And, it facilitates the realizations of collapse and revivals as well as nonclassical states of the ion motion, such as Fock, coherent, and squeezed states since one need not carefully arrange the ion such that it locates around the node of a standing wave. Also, the dressed-state Hamiltonian in interaction picture [Eq. (6)] might have other applications. For instance, it can be utilized to simplify the discussions and the expressions of cooling and heating rates when the decay rate Γ is taken into account, which is currently underway, and corresponding results will be published elsewhere. Here, we only give a brief discussion. From Eq. (6), one sees that the fluorescent spectrum will generally have three peaks centered at $\nu = 0, \pm\Omega_t$ where Ω_t is given by Eq. (5) when the decay rate Γ is omitted and it is given by $\Omega_t = \sqrt{\Omega_\phi^2 + \Delta^2 + 5\Gamma^2}$ when the decay rate is taken into account. The peak positions agree fairly well with the previous numerical computations [14]. From Eq. (6), one also sees that there is not the peak at $\nu = 0$ as $\phi = \pi/2$, and that the three peaks do not exist as $\phi = 0, \pi$. Again, these results also agree well with the previous numerical computations [14]. As $\phi = 0, \pi$, i.e., at the antinode of the standing wave, the coupling parameter in the interaction Hamiltonian of the JC model [Eq. (7)] becomes zero, and hence higher order terms describing two-phonon or multiphonon processes should be taken into account which will be discussed elsewhere.

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