Calculation of Hadron Form Factors from Euclidean Dyson-Schwinger Equations

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We apply Euclidean time methods to phenomenological Dyson-Schwinger models of hadrons. By performing a Fourier transform of the momentum space correlation function to Euclidean time and by taking the large Euclidean time limit, we project onto the lightest on-mass-shell hadron for given quantum numbers. The procedure, which actually resembles lattice gauge theory methods, allows the extraction of moments of structure functions, moments of light-cone wave functions, and form factors without *ad hoc* extrapolations to the on-mass-shell points. We demonstrate the practicality of the procedure with the example of the pion form factor. [S0031-9007(97)03002-0]

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The planned experimental program at the Thomas Jefferson National Accelerator Facility (TJNAF) will subject the electromagnetic structure of hadrons and nuclei to detailed scrutiny. The calculation of hadron form factors is therefore of fundamental importance to the subsequent interpretation of the obtained experimental results. However, the nonperturbative description of the hadron structure in the Minkowski metric, which has only recently been pursued in detail [1], is extremely difficult due to the direct confrontation with singularities and the indefinite norm. Alternatively, Euclidean space is characterized by a positive definite norm, i.e., $p^2 \equiv p_1^2 + p_2^2 + p_3^2 + p_4^2 \geq 0$, and it has long been known that the Euclidean formulation is therefore advantageous in the description of nonperturbative processes through the Dyson-Schwinger (DSE) and Bethe-Salpeter (BSE) equations.

These components are frequently assembled into diagrams such as that in Fig. 1 for the calculation of form factors [2,3]. The problem with the Euclidean formulation of such processes is that for physical particles the external momenta must satisfy the mass-shell condition $P^2 = -M_1^2$ and $K^2 = -M_2^2$, thus forcing the return to Minkowski space. One is then faced with the problem of complex momenta flowing through the loop in the diagram of Fig. 1. This in turn requires solving the DSE's in the complex plane. Although some progress has been made in this regard [4], it is an extremely difficult problem, and most calculations until now have employed entire-function fits on the real axis. Although some success has been obtained with this approach [3], there are uncertainties associated with the extrapolation of these functions into the complex plane.

Here we offer a fresh approach which avoids these uncertainties by obtaining the mass-shell conditions implicitly through an application of the Cauchy integral formula. (A similar method has been applied in Ref. [5] to 2-point functions, where it has been used as a means to obtain the physical mass. However, in this paper, we would like to demonstrate that Euclidean time projection methods are much more powerful and can be successfully applied to higher point functions as well.) This approach does not circumvent the need to make assumptions about the analytic structure of the components in the diagram of Fig. 1, but rather relies on the selection of relevant singularities in much the same way as is accomplished by Euclidean-space lattice calculations. We consider the pion form factor as a prototype for the purposes of illustration.

The exact calculation of the pion form factor via lattice techniques has been studied previously [6,7], and proceeds as follows: one first considers the Euclidean three-point correlation function

$$C_\mu(t_x, t_y) = \int d^3x\, d^3y \times \langle 0|T[J_\pi^\dagger(y) J_\mu(0) J_\pi(x)]|0\rangle e^{-i\vec{K}\cdot\vec{y}} e^{i\vec{P}\cdot\vec{x}}, \tag{1}$$

where $J_\pi(x) = \overline{u}(x) i\gamma_5 d(x)$ is the interpolating field for the pion and $J_\mu(0) = \frac{2}{3}\overline{u}(0)\gamma_\mu u(0) - \frac{1}{3}\overline{d}(0)\gamma_\mu d(0)$ is the vector current. t_x and t_y are the time components of x and y, respectively. Upon inserting a complete set of states between each pair of operators in Eq. (1) and taking the

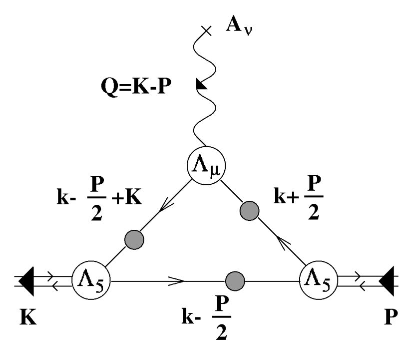

FIG. 1. The electromagnetic vertex for a composite pion at tree level. The quark Green's functions, quark-photon vertex, and the pion Bethe-Salpeter amplitudes, are dressed in a consistent, gauge-invariance manner.

limit $t_x \to -\infty$ and $t_y \to \infty$, so that only the lightest state contributes significantly, one finds

$$\lim_{t_y,-t_x\to\infty} C_\mu(t_x,t_y) = -\frac{e^{E_{\vec{P}}t_x}e^{-E_{\vec{K}}t_y}}{4E_{\vec{P}}E_{\vec{K}}}\langle 0|J_\pi^\dagger(0)|\pi,\vec{K}\rangle \times \langle \pi,\vec{K}|J_\mu(0)|\pi,\vec{P}\rangle\langle \pi,\vec{P}|J_\pi(0)|0\rangle\,, \tag{2}$$

where $|\pi,\vec{P}\rangle$ represents an on-mass-shell (i.e., $E_{\vec{P}} \equiv \sqrt{M_\pi^2 + \vec{P}^2}$) pion state with three-momentum $\vec{P}$. Note that Eq. (2) is exact. Thus, despite the Euclidean formulation (as long as one does not employ any further approximations such as quenching, finite quark masses, finite lattice spacing, finite volume), it allows an exact computation of the on-shell pion electromagnetic form factor

$$(\tilde{P}_\mu + \tilde{K}_\mu)F_\pi[(\tilde{P} - \tilde{K})^2] = \langle \pi,\vec{K}|J_\mu(0)|\pi,\vec{P}\rangle\,, \tag{3}$$

where $\tilde{P} \equiv (E_{\vec{P}},\vec{P})$ and $\tilde{K} \equiv (E_K,\vec{K})$. In practice [6] Eq. (1) is usually evaluated in the quenched approximation, yielding

$$C_\mu(t_x,t_y) = \int dU\, e^{-S(U)}\int d^3x\, d^3y\, e^{-i\vec{K}\cdot\vec{y}}e^{i\vec{P}\cdot x} \times \mathrm{tr}[\gamma_5 M^{-1}(U)\,(x,0)\gamma_\mu M^{-1}(U)\,(0,y) \times \gamma_5 M^{-1}(U)\,(y,x)]\,, \tag{4}$$

where $M^{-1}(U)\,(y,x)$ is the quark propagator for a given configuration of link fields U. Note that the group integration $\int dU$ implies that the quark lines and the vertices are dressed with arbitrary gluon lines.

The above example (plus many other examples in Euclidean lattice gauge theory) shows that, despite the fact that the calculations are performed in a Euclidean metric, it is still possible to extract on-mass-shell matrix elements, without having to make *ad hoc* assumptions about the analytic continuation back to a Minkowskian metric. The basic idea is that large Euclidean time merely acts as a filter to project out the ground state of the Hamiltonian. In this Letter, we present a procedure that uses similar techniques in the context of Euclidean Dyson-Schwinger calculations, which also there allows an unambiguous extraction of the on-mass-shell matrix elements.

The Dyson-Schwinger approach to form factor calculations.—An alternative to performing the exact (or quenched) calculation is to employ two approximations restricting the type of allowed gluon dressing. The first is to consider only dressing by the gluon two-point function, and the second is to consider only ladder/rainbow dressing. (Note that ladder/rainbow dressing implicitly leads to a restriction on allowed quark lines that resembles the quenched approximation in lattice QCD.) These approximations are in fact implied by the diagram of Fig. 1, and allow the separate calculation of the components, i.e., quark propagators and vertex functions. These approximations further comprise an electromagnetic gauge-invariant description [2,8]. Within this approximation the diagram of Fig. 1 is given by

$$\Lambda^5_\mu(K,P) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\Big[\Lambda_5(P,k)G\Big(k+\frac{P}{2}\Big)\Lambda_\mu\Big(K-P,k+\frac{K}{2}\Big) \times G\Big(k-\frac{P}{2}+K\Big)\Lambda_5\Big(K,k-\frac{P}{2}+\frac{K}{2}\Big)G\Big(k-\frac{P}{2}\Big)\Big]\,, \tag{5}$$

where Λ_5 is the solution of the *inhomogeneous* pseudoscalar BSE (quark-pseudoscalar vertex), Λ_μ is the solution of the *inhomogeneous* vector BSE (quark-photon vertex), and G is the quark propagator.

We will use the notation Λ to represent the solutions to the *inhomogeneous* BSE, and Γ to represent the solutions to the *homogeneous* BSE. In the vicinity of a pole, they are related by [8]

$$\Lambda(P,k) \approx \frac{M^2\sqrt{Z(-M^2)}}{P^2+M^2}\,\Gamma(P,k)\,, \tag{6}$$

where M is the pole mass, $Z(-M^2)$ is the wave function renormalization, and $\Gamma(P,k)$ therefore obeys the standard Bethe-Salpeter bound state normalization [8].

The vertex function for a photon coupling to an on-shell pion is then given by

$$\Gamma^\pi_\mu(K,P) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\Big[\Gamma_\pi(P,k)G\Big(k+\frac{P}{2}\Big)\Lambda_\mu\Big(K-P,k+\frac{K}{2}\Big) \times G\Big(k-\frac{P}{2}+K\Big)\Gamma_\pi\Big(K,k-\frac{P}{2}+\frac{K}{2}\Big)G\Big(k-\frac{P}{2}\Big)\Big]\,, \tag{7}$$

wherein the vicinity of the pion pole

$$\Lambda^5_\mu(K,P) \approx \frac{M_\pi^4 Z(-M_\pi^2)}{(P^2+M_\pi^2)(K^2+M_\pi^2)}\,\Gamma^\pi_\mu(K,P)\,. \tag{8}$$

The quantity $\Gamma^\pi_\mu(K,P)$ has the decomposition [2]

$$\Gamma^\pi_\mu(K,P) = (P_\mu + K_\mu)F_\pi(P^2,K^2,(K-P)^2) + (K_\mu - P_\mu)(P^2 - K^2) \times H_\pi(P^2,K^2,(K-P)^2), \quad (9)$$

and is commonly used to extract the pion form factor, F_π, at the on-shell point $P^2 = K^2 = -M_\pi^2$.

Rather than explicitly enforcing the mass-shell condition as has been explored previously [3], here we instead take an approach that is similar to that taken in the Euclidean-space lattice calculation. The quantity in the DSE approach that is analogous to the three-point correlation function, $C_\mu(t_x,t_y)$ in (4), is

$$C'_\mu(t_x,t_y) \equiv -\int_{-\infty}^{\infty} \frac{dP_4 dK_4}{(2\pi)^2}\, e^{-iP_4 t_x} e^{iK_4 t_y} \Lambda^5_\mu(K,P)\,. \quad (10)$$

The strength of this approach is that the k-integration necessary to evaluate Eq. (7) is now over real functions, functions that can be determined from the Euclidean-space Dyson-Schwinger equations. In practical applications, both the integration in Eq. (10) as well as in Eq. (7) are performed numerically.

The form factor is obtained implicitly from the expression

$$\lim_{t_y,-t_x\to\infty} C'(t_x,t_y) = -\frac{e^{E_{\vec P}t_x}e^{-E_{\vec K}t_y}}{4E_{\vec P}E_{\vec K}}\,[M_\pi^2\sqrt{Z(-M_\pi^2)}\,] \times [(\tilde P_\mu + \tilde K_\mu)F_\pi((\tilde K - \tilde P)^2)] \times [M_\pi^2\sqrt{Z(-M_\pi^2)}\,]. \quad (11)$$

The correctness of Eq. (11) is readily verified by substituting Eqs. (8) and (9) into (10), along with the implicit assumption that there are no poles in the immediate vicinity of the ground-state pion pole of interest. The validity of this assumption is, of course, model dependent. Equation (11) can be directly compared with the exact result in Eq. (2).

Toy-model calculation.—In principle, Eq. (11) illustrates how the form factor is extracted from the 3-point function for large Euclidean times. It remains to be demonstrated that the procedure is practical. For this purpose, we now examine a toy model where all integrals can be performed analytically to yield an "exact" reference calculation. The model is then also treated numerically in exactly the same way as one would proceed in the general case where the Bethe-Salpeter amplitudes and vertex functions cannot be treated analytically. We are thus able to get some idea about typical numerical requirements of our method.

We now proceed to evaluate Eq. (11) for a simple model. Here we make the following choices

$$\Lambda_5(P,k) \equiv \frac{i\gamma_5}{P^2}\,, \qquad \Lambda_\mu(P,k) \equiv -i\gamma_\mu\,, \qquad G(k) \equiv \frac{1}{i\gamma k + m_q}\,, \quad (12)$$

where m_q is taken to be a constituent quark mass. Above, in Eq. (6), we introduced a factor of M^2 into the definition of the vertex function, since this makes the vertex function dimensionless. However, in the strict $M^2 = 0$ case, this would lead to ill-defined expressions and we thus leave out the factor M^2 in this toy model.

Equation (12) implies in particular that $\Gamma_\pi = i\gamma_5$. The exact result can be obtained analytically by directly fixing the mass-shell condition in the evaluation of Eq. (7) and Eq. (9), and is shown by the solid line in Fig. 2. The constituent quark mass, m_q, is taken to be 300 MeV, and a cutoff of 1 GeV^2 is applied on the relative four-momentum^2 at the quark-photon vertex [i.e., $(k + K/2)^2$ in Fig. 1] to regulate the integration. (This cutoff is necessary for the toy model, but is not a general feature of the technique that we are using. For instance, in a more realistic calculation, the Bethe-Salpeter amplitudes would cut off the integration naturally.) The external momenta are constrained such that $P^2 = K^2 = 0$.

For comparison, we evaluate Eq. (11) by substituting the toy model Eq. (12) into Eq. (5) for the vertex Λ^5_μ. The results are shown by the series of dashed curves in Fig. 2 for time separations $\Delta t \equiv t_y - t_x \sim 2$ fm. The Fourier transform is performed with 4096 equally spaced points with a grid spacing of 0.02 GeV. Note that the agreement with the exact result diminishes at larger momentum transfers Q^2, as is expected. The agreement at large momentum is improved by increasing the number of Fourier transform points to include larger momentum components in the integration.

There is an additional effect due to the presence of poles in the quark propagators. This is revealed in Fig. 3 where, for example, a constituent quark mass of 1 GeV

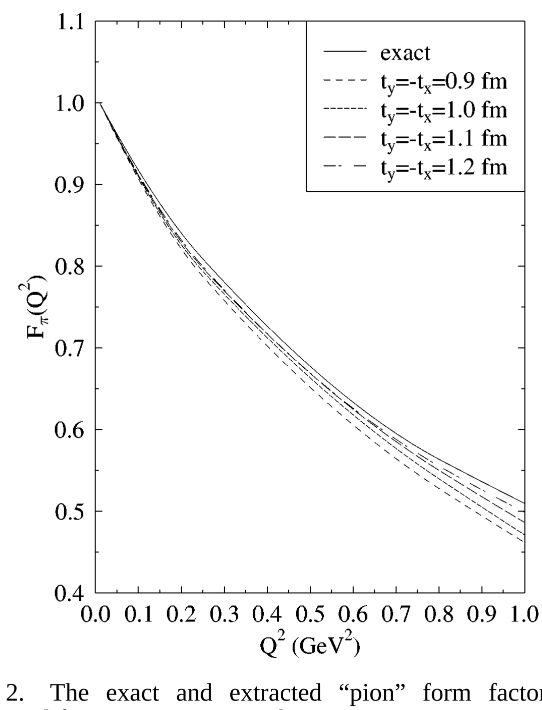

FIG. 2. The exact and extracted "pion" form factors are compared for a constituent quark mass $m_q = 300$ MeV.

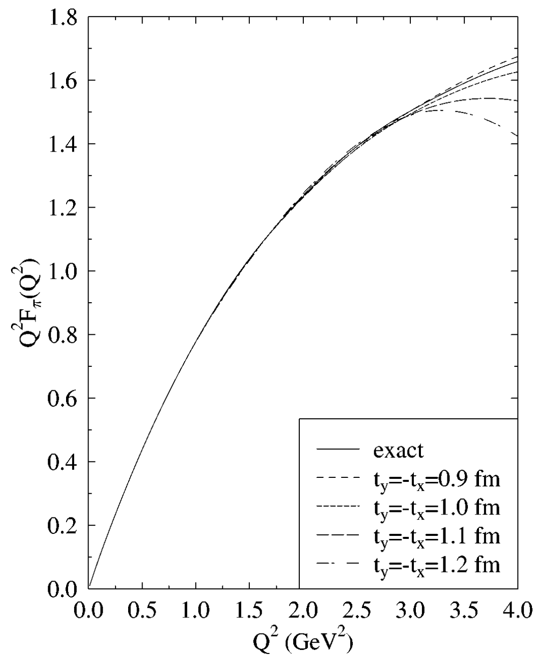

FIG. 3. The exact and extracted "pion" form factors are compared for a constituent quark mass $m_q = 1.0$ GeV.

is used with 1024 Fourier transform points at a grid spacing of 0.03 GeV. With this large quark mass the associated singularities are strongly damped and therefore do not contaminate the extraction of the pion pole. The comparison with the exact result is now quite good out to reasonably large momenta. The increasing discrepancy with increasing time separation at large momenta is due to the highly oscillatory nature of the Fourier transform, and is improved with a finer grid spacing. It is expected that the use of quark propagators from DSE studies [9] that reflect the confining nature of QCD will also produce accurate results at higher momenta due to the absence of such poles.

We have introduced Euclidean time projection methods as a means to project Euclidean metric 3-point functions onto the physical mass shell. The method, which is based on Cauchy's theorem and the existence of a Lehmann representation, avoids having to make *ad hoc* assumptions about the analytic behavior of propagators and vertex functions, and thus allows the direct use of Euclidean DSE's for the calculation of form factors. We have demonstrated the feasibility of the method by studying the pion form factor for a toy model where the exact form factor is known. The approach resembles methods that have been used in the context of lattice gauge theory [6,7]. The idea is to first Fourier transform n-point functions from (Euclidean) energy to (Euclidean) time. The Euclidean time difference between hadron interpolating fields and currents that are used to "probe" the hadron structure is then taken to be large. This ensures that the hadron that is being probed is actually the ground state with the quantum numbers of the interpolating field.

Even though we have studied the pion form factor as a showcase example, applications of Euclidean time projection methods should be applicable to a much wider class of physical observables. Starting with 2-point functions, one can obtain masses [5] as well as light-cone moments of hadron wave functions [7] on the mass shell. Examples for 3-point functions that we plan to calculate include the elastic form factor, moments of parton distribution functions (e.g., momentum or spin fraction carried by the quarks in a given hadron), and the $\gamma^*\pi \rightarrow \gamma$ transition form factor. The most simple observables related to 4-point functions that one might consider calculating using Euclidean time projection are hadron polarizabilities and (virtual) Compton scattering processes.

The main limitations of the method are first of all the restriction to the lightest hadrons for given quantum numbers; though theoretically possible, it is in practice rather difficult to use the method to obtain observables that involve excited states, since the whole trick is to use large Euclidean times to suppress excited states (a limitation that is very familiar to the lattice gauge community). Another limitation concerns very large momentum transfers, where rapid oscillations of the integrand require a large number of integration points. However, even with rather moderate computing power, this still allows access to most of the kinematic range that is available at TJNAF.

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