Landau Level Mixing and Levitation of Extended States in Two Dimensions

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(Received 21 June 1996)

We study the effects of mixing of different Landau levels on the energies of one-body states, in the presence of a strong uniform magnetic field and a random potential in two dimensions. We use a perturbative approach and develop a systematic expansion in both the strength and smoothness of the random potential. We find the energies of the extended states shift *upward,* and the amount of levitation is proportional to $(n + 1/2)/B^3$ for strong magnetic field, where B is the magnetic field strength and n is the Landau level index. [S0031-9007(96)02102-3]

PACS numbers: 71.30.+h, 73.40.Hm

The behavior of extended electronic states of noninteracting electrons in a uniform magnetic field B with a random substrate potential $V(\mathbf{r})$ is of central importance to the understanding of the integer quantum Hall effect [1]. In this Letter, we report a new and rather simple calculation that exposes the microscopic origin of the so-called "levitation" of extended states [2,3] in the large-B limit, which has been the subject of recent interest.

On the one hand, it is now widely accepted that, in the limit $B = 0$, there are no extended (delocalized) single-electron states at any finite energy [4], while in the strong-field limit, there exist discrete energies near the center of each disorder-broadened Landau level, at which states are extended [5–8]. An appealing (but heuristic) scenario, known as the "levitation" of extended states, has been proposed to explain how the interpolation between these limiting behaviors might occur [2,3]. This holds that one-electron states are localized at all energies except at a discrete set $E_n^c(B) = (n + \frac{1}{2})\hbar\omega_c + \epsilon_n(B)$, $n \geq 0$, where $\omega_c = |eB|/m$. The energies $\epsilon_n(B) \rightarrow \epsilon_c$, a constant, as $|B| \rightarrow \infty$, and increase monotonically as $|B|$ decreases, in such a way that $E_n^c(B) < E_{n+1}^c(B) < \infty$ for $|B| > 0$, finally diverging as $B \rightarrow 0$. This scenario is the basis of the recently proposed global phase diagram for the quantum Hall effect [9]. We emphasize the levitation, $\epsilon_n(B)$, is defined *relative* to the Landau level energy $(n + \frac{1}{2})\hbar\omega_c$, which depends linearly on B.

While the levitation scenario is appealing, it has apparently not yet been derived from microscopic considerations, and recently there has been considerable interest in testing it experimentally and numerically [10–19], and in identifying its microscopic origin. The effect must be associated with Landau-level mixing, which gives rise to an apparent paradox: generically, mixing gives rise to a *level-repulsion* effect, which would tend to *lower* rather than raise the energy levels. (This is clear for the case $n = 0$, but is generally true, as the level repulsion due to mixing with higher Landau levels is always stronger than that from lower ones.)

In this Letter, we resolve this paradox, and provide a rather simple explanation of the initial appearance of levitation associated with Landau-level mixing at large but finite fields, giving the $O(B^{-3})$ levitation (relative to the Landau level energy)

$$\epsilon_n(B) = \epsilon_c + (n + \tfrac{1}{2})\hbar\omega_c\left(\frac{(\ell/\xi)^2}{\omega_c\tau}\right)^2 + O(B^{-4}), \quad (1)$$

where $\ell = \sqrt{\hbar/|eB|}$ is the "magnetic length." Here $\hbar/\tau$ is the energy scale of Landau-level broadening in the high-field limit [essentially the variance of the fluctuations of $V(\mathbf{r})$], and ξ is a characteristic length scale over which the potential varies by this amount. This result is derived in the limit $\ell/\xi \ll 1$ and $\omega_c\tau \gg 1$, which is always achieved at sufficiently high magnetic fields provided the potential is bounded, local, and smoothly varying.

The basic idea that leads to the above conclusion is summarized in the following three paragraphs. In the limit $\hbar\omega_c \rightarrow \infty$ [20], i.e., the spacing between Landau levels is infinite, mixing between different Landau levels is not allowed, and one can work in the *truncated Hilbert space* of a given Landau level. The problem of localization in an isolated Landau level has been studied extensively [5–8,21] and the physics is very well understood, especially when the potential is smooth. In this limit the Landau quantization becomes exact, the dynamics of cyclotron and "guiding center" motions of electrons decouple, and the latter can be treated semiclassically: the electrons move adiabatically along equipotentials of the potential $V(\mathbf{r})$, with the local drift velocity, $\mathbf{v}_d = \hat{\mathbf{z}} \times \nabla V(\mathbf{r})/eB$, where $\hat{\mathbf{z}}$ is the direction of the magnetic field. Trugman [21] (see also Ref. [22]) pointed out that in this limit the delocalization of electronic states is associated with the percolation of equipotential lines, and the energy at which the equipotential lines percolate is determined by the potentials at the saddle points, especially the *critical saddle point:* $\epsilon_c = V(\mathbf{r}_c)$, where $\mathbf{r}_c$ is the location of the critical saddle point. Corrections to this semiclassical behavior will occur when the equipotential line on which a particle is moving comes close to a saddle point of $V(\mathbf{r})$, and tunneling to a nearby equipotential line at the same energy can occur [23]. This breakdown is believed to control the quantum critical behavior when ϵ is close to ϵ_c [5]. The

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energy of the extended state is thus clearly determined by saddle-point potentials, in this limit.

When $\hbar\omega_c$ is large but finite, there is finite mixing between different Landau levels, and working in a truncated Hilbert space with the original (bare) potential misses effects due to such mixing. We develop in the following a formalism, which allows us to continue working in the truncated Hilbert space, and in the mean time taking into account all Landau level mixing effects by *renormalizing* the *effective* potential seen by the electrons in a given Landau level. This *locally renormalized Landau-level-dependent effective potential* may be calculated perturbatively. We may calculate any physical quantities in the *truncated* Hilbert space, using the *renormalized* potential, and the results should be the same as obtained from the calculation in the *full* Hilbert space, using the *bare* potential. We will demonstrate the validity of our approach by applying it to the specific model of a purely quadratic saddle-point potential, which has been treated by Fertig and Halperin [23] exactly. We emphasize, however, our approach may be applied to *any* potential that is bounded and smooth.

We find the renormalized effective potential has the following feature: although it is renormalized *downward* almost everywhere due to level repulsion effects, the renormalization at *saddle points* is nevertheless *positive definite,* leading to levitation of extended states.

We find the locally renormalized Landau-level-dependent effective potential takes the following form:

$$V_{\mathrm{eff}}^{(n)}(\mathbf{r}) = V(\mathbf{r}) + \sum_{m\geq 2} V_m^{(n)}(\mathbf{r}), \tag{2}$$

where $V_m^{(n)}(\mathbf{r}) \propto B^{-m}$ as $B \to \infty$. The leading $O(B^{-2})$ correction is given by

$$V_2^{(n)}(\mathbf{r}) = -\frac{\ell^2}{2\hbar\omega_c}|\nabla V(\mathbf{r})|^2 \leq 0, \tag{3}$$

which is independent of the Landau level index, and negative. This is the generically dominant "level-repulsion" term, which indeed causes a downward shift of typical energy levels. It is proportional to the square of the local electric field strength, and is the only correction in the trivially solvable case where the substrate potential $V(\mathbf{r})$ is that of a uniform electric field.

The crucial observation is that $V_2^{(n)}(\mathbf{r})$ *vanishes* at *all* saddle points. Since ϵ_n is determined by the potential of saddle points, the dominant correction $V_2^{(n)}(\mathbf{r})$ *does not* affect the extended state energies.

The next order correction is

$$V_3^{(n)}(\mathbf{r}) = \tfrac{3}{8}(n + \tfrac{1}{2})\left(\frac{\ell^4}{\hbar\omega_c}\right)u(\mathbf{r}), \tag{4}$$

where

$$\begin{aligned} u(\mathbf{r}) &= [\nabla^2 V(\mathbf{r})]^2 - \det_{ij}|\nabla_i\nabla_j V(\mathbf{r})|, \\ &\equiv (\nabla_x^2 V - \nabla_y^2 V)^2 + (2\nabla_x\nabla_y V)^2 \geq 0. \end{aligned} \tag{5}$$

At a saddle point $u(\mathbf{r}) > 0$, as the determinant of second derivatives is negative. Thus, in contrast to a generic point where the corrections to the effective potential due to Landau-level mixing are negative, the leading correction at saddle points, which control the energies of extended states, is *positive*, giving rise to the levitation effect. The result (1) follows from an estimate of $u(\mathbf{r})$ at saddle points as being of order $(\hbar/\tau\xi^2)^2$. Our results are schematically illustrated in Fig. 1.

We now sketch the technical derivation of (2)–(5). We write the substrate potential $V(\mathbf{r})$ in terms of its Fourier components $\tilde{V}(\mathbf{q})$

$$V(\mathbf{r}) = \frac{1}{A}\sum_{\mathbf{q}} \tilde{V}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{6}$$

where for convenience we have imposed (quasi-)periodic boundary conditions on an area A that contains an integral number of magnetic flux quanta. We now write

$$e^{i\mathbf{q}\cdot\mathbf{r}} = e^{i\mathbf{q}\cdot\mathbf{R}}U(\mathbf{q}), \qquad U(\mathbf{q}) = e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{R})}, \tag{7}$$

where $\mathbf{R}$ is the "guiding center" of the cyclotron orbit [24], which obeys the algebra [24,25]

$$e^{i\mathbf{q}\cdot\mathbf{R}}e^{i\mathbf{q}'\cdot\mathbf{R}} = \exp[\tfrac{1}{2}i(\mathbf{q}\times\mathbf{q}')\ell^2]e^{i(\mathbf{q}+\mathbf{q}')\cdot\mathbf{R}}. \tag{8}$$

(Here $\mathbf{q}\times\mathbf{q}' \equiv q_x q_y' - q_y q_x'$.) The unitary operator $U(\mathbf{q})$ acts entirely on the cyclotron orbit (Landau level) variables, and commutes with the guiding center. In the strong-field limit, the potential term projected into the Landau level n becomes

$$\frac{1}{A}\sum_{\mathbf{q}} \tilde{V}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{R}}U(\mathbf{q})_{nn}, \tag{9}$$

where $U(\mathbf{q})_{nn'} \equiv \langle n|U(\mathbf{q})|n'\rangle$ (n and n' are Landau-level indices): for $n \geq n'$, $U_{nn'}(\mathbf{q})$ is given by

$$\left(\frac{(q_x + iq_y)\ell}{\sqrt{2}}\right)^{n-n'} L_{n'}^{n-n'}(\tfrac{1}{2}q^2\ell^2)\exp(-\tfrac{1}{4}q^2\ell^2), \tag{10}$$

where $L_n^m(x)$ is a Laguerre polynomial.

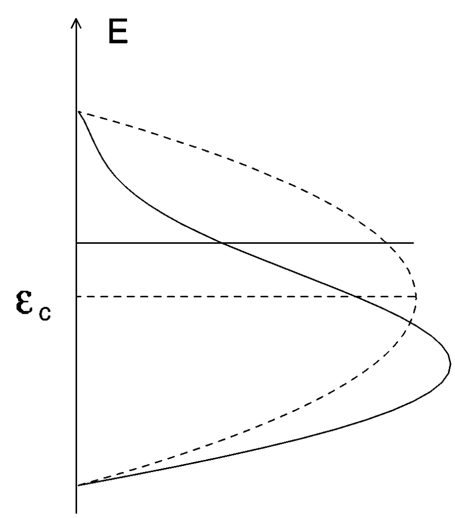

FIG. 1. Density of states and energy of extended states in a given Landau level before (dashed lines) and after (solid lines) Landau level mixing is taken into account.

The problem in the high-field limit is to diagonalize the projected potential (9), in the subspace of a given Landau level. When the field strength is strong but finite, states in different Landau levels are still *well separated.* Nevertheless, electrons in a given Landau level may be scattered into other Landau levels by the potential, and will eventually come back due to energy conservation. The effect of such (virtual) processes is to *renormalize* the *effective* potential seen by the electrons in this Landau level [see Fig. 2], which we calculate below.

The trick we will use to characterize the renormalization is to develop a perturbative expansion in $V/\hbar\omega_c$, and rewrite the effective Hamiltonian in the form (9), but with a renormalized $\tilde{V}_{\rm eff}^{(n)}(\mathbf{q})$, which can then be expanded in powers of ℓ as well as in $1/\hbar\omega_c$, to give a true $1/B$ expansion. We then carry out the Fourier transform to find the renormalized $V_{\rm eff}^{(n)}(\mathbf{r})$ that this corresponds to.

Using standard perturbative renormalization formalism, we find the leading $O(V^2/\hbar\omega_c)$ term in the effective Hamiltonian is

$$\frac{1}{A^2}\sum_{\mathbf{q}\mathbf{q}'}\frac{\tilde{V}(\mathbf{q})\tilde{V}(\mathbf{q}')}{\hbar\omega_c}e^{i\mathbf{q}\cdot\mathbf{R}}e^{i\mathbf{q}'\cdot\mathbf{R}}\sum_{n'}{}'\,\frac{U_{nn'}(\mathbf{q})U_{n'n}(\mathbf{q}')}{(n-n')}\,. \tag{11}$$

The primed sum means that the singular term $n' = n$ is excluded. We must now express this term in the form (9), using the contraction (8). The general $O[V^m/(\hbar\omega_c)^{m-1}]$ contribution to $V_{\rm eff}^{(n)}(\mathbf{r})$ may be written (for $m > 1$) in the form

$$\frac{\hbar\omega_c}{A^m}\sum_{\mathbf{q}_1\ldots\mathbf{q}_m}\left(\prod_{i=1}^{m}\frac{\tilde{V}(\mathbf{q}_i)e^{i\mathbf{q}_i\cdot\mathbf{r}}}{\hbar\omega_c}\right)f_m^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_m)\,, \tag{12}$$

where $f_m^{(n)}(\mathbf{q}_1,\ldots,\mathbf{q}_m)$ is a *symmetric* and *analytic* function of the $\{\mathbf{q}_i\ell\}$ [it is derived from the $U_{nn'}(\mathbf{q})$, which are analytic]. It is also *rotationally invariant*, and must *vanish as any of the* $\mathbf{q}_i \to 0$, as addition of a spatially constant term (a $\mathbf{q} = 0$ Fourier component) to the potential cannot affect the nonlinear terms in $V_{\rm eff}^{(n)}(\mathbf{r})$. The term $f_2^{(n)}(\mathbf{q}_1,\mathbf{q}_2)$ is the symmetric part of

$$\frac{e^{i\mathbf{q}_1\times\mathbf{q}_2\ell^2/2}}{U_{nn}(\mathbf{q}_1+\mathbf{q}_2)}\sum_{n'}{}'\,\frac{U_{nn'}(\mathbf{q}_1)U_{n'n}(\mathbf{q}_2)}{(n-n')}\,. \tag{13}$$

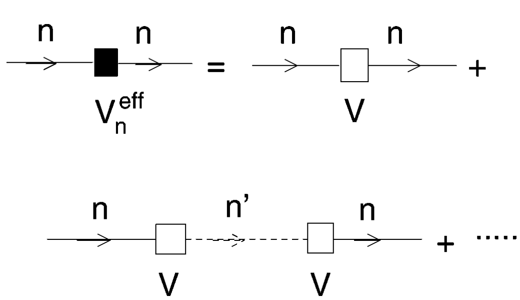

FIG. 2. Schematic perturbative expansion of the effective potential seen by electrons in the nth Landau level.

It is straightforward to expand $f_2^{(n)}(\mathbf{q}_1,\mathbf{q}_2)$ in powers of ℓ, using (10); we find that, up to terms of order ℓ^4, it is given by

$$\tfrac{1}{2}(\mathbf{q}_1\cdot\mathbf{q}_2)\ell^2 + \tfrac{3}{8}(n+\tfrac{1}{2})[(\mathbf{q}_1\cdot\mathbf{q}_2)^2 - (\mathbf{q}_1\times\mathbf{q}_2)^2]\ell^4. \tag{14}$$

This corresponds to a gradient expansion of the effective potential in real space, and gives the leading terms of $O(B^{-2})$ and $O(B^{-3})$ in (2).

We find that the leading term in the gradient expansion of the term of order $O[V^3/(\hbar\omega_c)^2]$ is of order ℓ^4 [this, in fact, follows directly from the general properties of $f_3^{(n)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ mentioned above]. This means that its leading contribution to the effective potential is $O(B^{-4})$, and it does not contribute to the leading terms. Higher-order terms in $V/\hbar\omega_c$ vanish even faster at large B.

Fertig and Halperin [23] (FH) considered the quantum mechanical problem of tunneling through a *quadratic* saddle-point potential, $V(x,y) = V_0 - U_x x^2 + U_y y^2$, which they were able to treat exactly. By applying our approach to this specific potential, one may check the validity of our results against the exact solution, which we illustrate below. For particles which are asymptotically in the nth Landau level, FH calculated the transmission coefficient and found the energy E_n^* at which reflection and transmission coefficients are equal. Expansion of their result in powers of $1/\hbar\omega_c$ gives

$$\begin{aligned}E_n^* &= V_0 + (n+\tfrac{1}{2})[\hbar\omega_c + (U_y - U_x)\ell^2]\\ &\quad + (n+\tfrac{1}{2})\frac{\ell^4}{\hbar\omega_c}[2U_xU_y - \tfrac{1}{2}(U_x-U_y)^2]\\ &\quad + O\big((\hbar\omega_c)^{-2}\big)\,.\end{aligned} \tag{15}$$

In the limit $\hbar\omega_c \to \infty$, where Landau levels decouple, we may write $E_n^* = (n+\frac{1}{2})\hbar\omega_c + \epsilon_n^*$, where

$$\epsilon_n^* = V(\mathbf{x}_c) + (n+\tfrac{1}{2})\tfrac{\ell^2}{2}\nabla^2V(\mathbf{x}_c) + O(\ell^4) \tag{16}$$

[where the correction terms $O(\ell^4)$ and above are absent for the purely quadratic saddle-point potential of [23]]. The formula (15) is exactly reproduced when (16) is evaluated using the *renormalized* (and no longer purely quadratic) potential $V_{\rm eff}^{(n)}(\mathbf{r})$. Thus our results are fully consistent with those of Ref. [23], as we are able to reproduce the exact result using a formula that is valid in the *truncated* Hilbert space, *and renormalized* potential. Such tunneling effects, however, do not affect our conclusion (1) since the configurationally averaged levitation of E_n^* has the same dependence on n and B as that of $V_{\rm eff}^{(n)}(\mathbf{x}_c)$.

We emphasize that the perturbative approach we use here is valid only when the magnetic field is strong, different Landau levels are well separated, and Landau level mixing is weak. In this regime our results clearly support the levitation scenario [2,3]. Nevertheless in this regime the quantized Hall conductance *increases,* as B decreases, because the decreasing of Landau level energies dominates the levitation effects. Other possible scenarios [16] at *weak* magnetic field, however, are not ruled out here.

In the following we discuss the experimental implications of our results, and their relation with existing work.

Our result shows that the leading effect in this limit is an $O(B^{-2})$ *downwards* motion of the mean energy of the Landau level, while the extended state is *static* to this order, and only levitates to $O(B^{-3})$. In this limit at least, experimental evidence [11–13] that the extended state rises relative to the mean energy of the Landau level would be demonstrating not levitation of extended states, but the lowering of localized state energies due to level repulsion between Landau levels. We also note that evidence of levitation of extended states has been found in previous numerical work, in both the continuum system [8,10], and the tight binding model [17], although there is controversy in the latter case [16,18].

Recently Shahbazyan and Raikh [14] (see also Ref. [15]) used an extension of the network model [5] to simulate the continuum system in the presence of a smooth random potential. They considered the effects of strongly localized orbitals of different Landau levels with energies close to the saddle-point energies of a particular Landau level, and find that resonant tunneling into such orbitals results *on average* in a reduction of the transmission rate through the saddle points, implying an upward shift of the energy of extended states. We note that in order for this effect to be important, there must be significant overlap in the density of states (DOS) of different Landau levels; while it is clear from our results that levitation occurs even if there is no overlap in the DOS of different Landau levels (which is the case when B is large). Later, Gramada and Raikh [19] studied the effects of a short-range impurity potential on the transmission rate through a nearby saddle point, and again find a reduction of the transmission rate *on average*. They estimate the upward shift of the extended state energy due to this effect to be of order B^{-4} for large B. We believe the $O(1/B^3)$ levitation we identify here is the dominant one, at large B.

There are recent observations [12,26] of apparently direct transitions from quantum Hall states with large ν to insulating states at very *weak* magnetic field, which appears to be inconsistent with the conventional one-electron extended-state-levitation picture and the global phase diagram [9]. We note at very weak magnetic field electron-electron interactions may become important and the one-body picture may not be sufficient. However, a quantitative validation of the levitation scenario for noninteracting electrons in the $B \to 0$ limit clearly urgently needs to be attempted.

To summarize: We have used a perturbative approach to study the effects of mixing between Landau levels in a two-dimensional noninteracting electron system, due to a random substrate potential. In high magnetic fields, we find that although most of the states in the Landau level with index n are pushed to lower energy by such mixing, the energy of extended states shifts upward, and the amount of this shift is proportional to $(n + 1/2)/B^3$.

We thank R. N. Bhatt, H. A. Fertig, S. M. Girvin, D. Shahar, and S. L. Sondhi for helpful discussions. This work was supported by NSF Grant No. DMR-9400362.

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