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Direct Construction of Conservation Laws from Field Equations

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This Letter presents an algorithm to obtain all local conservation laws for any system of field equations. The algorithm uses a formula which directly generates the conservation laws and does not depend on the system having a Lagrangian formulation, in contrast to Noether's theorem which requires a Lagrangian. Several examples are considered including dissipative systems inherently having no Lagrangian. [S0031-9007(97)02768-3]

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Finding conservation laws is important in the study of physical systems. Given a system of field equations arising from a Lagrangian, one can obtain all local conservation laws of the system algorithmically by using Noether's theorem. The algorithm involves the following steps: (1) Find all local symmetries of the given Lagrangian system [1,2]. (2) Check which of the symmetries leave the Lagrangian invariant to within a local divergence. (3) For each such symmetry construct a conservation law through the variational relation between the Lagrangian and the field equations [1,2]. However, given a system without a Lagrangian formulation, one currently lacks a corresponding algorithm to find local conservation laws of the system.

In this Letter we present an algorithm to obtain all local conservation laws for any system of field equations whether or not the system has a Lagrangian formulation. The algorithm uses an adjoint invariance condition together with a formula which directly yields a conservation law for any solution of the condition, and involves the following steps: (1) Linearize the given system of field equations and find the adjoint system of the linearized system. (2) Find all solutions of the adjoint system [3]. (3) Check which of the solutions satisfy our adjoint invariance condition. (4) For each such solution construct a conservation law directly by our formula.

The linearized system and the adjoint system are the same if and only if the given system is self-adjoint, in which case the system has a Lagrangian formulation. For a Lagrangian system the solutions of the linearized system are simply the symmetries of the field equations, and our adjoint invariance condition is equivalent to the condition that a symmetry leaves the Lagrangian invariant to within a local divergence. However, the construction of conservation laws for such a system is considerably simpler using our algorithm than using Noether's theorem since the adjoint invariance condition and conservation law formula do not require the expression for the local divergence arising from invariance of the Lagrangian, which is a cumbersome yet essential step in Noether's theorem.

In our algorithm, for any given system of field equations, the formula used for the construction of conservation laws is well-defined for an arbitrary solution of the adjoint system. Consequently, after steps (1) and (2), one can bypass step (3) and simply insert each solution of the adjoint system into our conservation law formula, then check whether the resulting expression is conserved for all solutions of the field equations. (This provides a useful shortcut in practice, since invariably all solutions of the adjoint system with the exception of scaling solutions satisfy the adjoint invariance condition.)

Consider any given system of field equations [4]

$$G_\Omega[u] = 0 \tag{1}$$

for field variables $u^\sigma(x^i)$, with time and space coordinates represented by variables x^i. The linearization of system (1) is given by

$$\mathcal{L}_{\Omega\rho}[u]\eta^\rho = 0\,, \tag{2}$$

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where

$$\mathcal{L}_{\Omega\rho}[u] = G_{\Omega\rho}[u] + G^{i}_{\Omega\rho}[u]D_i + G^{ij}_{\Omega\rho}[u]D_iD_j + \cdots, \tag{3}$$

with D_i denoting the coordinate derivative operator with respect to x^i, and $G_{\Omega\rho}[u] = \partial G_\Omega[u]/\partial u^\rho$, $G^i_{\Omega\rho}[u] = \partial G_\Omega[u]/\partial D_i u^\rho$, etc. The adjoint system of system (2) is given by

$$\mathcal{L}^*_{\rho\Omega}[u]\Lambda^\Omega = G_{\Omega\rho}[u]\Lambda^\Omega - D_i(G^i_{\Omega\rho}[u]\Lambda^\Omega) + D_iD_j(G^{ij}_{\Omega\rho}[u]\Lambda^\Omega) + \cdots = 0, \tag{4}$$

with $\mathcal{L}^*_{\rho\Omega}[u]$ defining the formal adjoint of $\mathcal{L}_{\rho\Omega}[u]$. In systems (2) and (4), u^σ is an arbitrary solution of the field equations (1), while η^σ and Λ^Ω are variables depending *locally* on u^σ and derivatives of u^σ.

Our adjoint invariance condition on a solution $\Lambda^\Omega[u]$ of system (4) is given by

$$\mathcal{L}^*_{\sigma\Omega}[U]\Lambda^\Omega[U] = -\Lambda^\Omega_\sigma[U]G_\Omega[U] + D_i(\Lambda^{\Omega i}_\sigma[U]G_\Omega[U]) - D_iD_j(\Lambda^{\Omega ij}_\sigma[U]G_\Omega[U]) + \cdots \tag{5}$$

for *arbitrary* $U^\sigma(x^i) = u^\sigma(x^i)$ which are field variables not subject to the field equations, where $\Lambda^\Omega_\sigma[U] = \partial\Lambda^\Omega[U]/\partial U^\sigma$, $\Lambda^{\Omega i}_\sigma[U] = \partial\Lambda^\Omega[U]/\partial D_i U^\sigma$, etc. Given any $\Lambda^\Omega[u]$ satisfying condition (5), we have the following conservation law on all solutions u^σ of the field equations (1):

$$D_i\Phi^i[u] = 0, \tag{6}$$

where [5]

$$\Phi^i[u] = \int_0^1 \frac{d\lambda}{\lambda}\,(S^i[U] + N^i_\sigma[U]U^\sigma + N^{ij}_\sigma[U]D_jU^\sigma + \cdots)|_{U=\lambda u}, \tag{7}$$

$$S^i[U] = \Lambda^\Omega[U]G^i_{\Omega\sigma}[U]U^\sigma + (\Lambda^\Omega[U]G^{ij}_{\Omega\sigma}[U]D_jU^\sigma - D_j(\Lambda^\Omega[U]G^{ij}_{\Omega\sigma}[U])U^\sigma) + \cdots, \tag{8}$$

$$N^i_\sigma[U] = \Lambda^{\Omega i}_\sigma[U]G_\Omega[U] - D_j(\Lambda^{\Omega ij}_\sigma[U]G_\Omega[U]) + \cdots, N^{ij}_\sigma[U] = \Lambda^{\Omega ij}_\sigma[U]G_\Omega[U] - D_k(\Lambda^{\Omega ijk}_\sigma[U]G_\Omega[U]) + \cdots.$$

The proof that the expression $\Phi^i[u]$ satisfies Eq. (6) is left to a forthcoming paper [6], where we also show that for any given system (1) *all* local conservation laws are obtained by our algorithm.

To illustrate our adjoint invariance condition (5) and conservation law formula (7), we now consider several examples of scalar field equations in two spacetime dimensions. For the sequel, $x^0 = t$, $x^1 = x$ denote time and space coordinates, $D_0 = D_t$, $D_1 = D_x$ denote coordinate derivatives, and $u(x, t)$ denotes a scalar field.

Nonlinear wave equation.—Consider a Lagrangian $L[u] = -\frac{1}{2}(D_tu)^2 + \frac{1}{2}(D_xu)^2 + (p+1)^{-1}u^{p+1}$ for any $p > 1$. The field equation arising from $L[u]$ is given by

$$G[u] = D_t^2u - D_x^2u + u^p = 0. \tag{9}$$

Linearizing Eq. (9) leads to the operator

$$\mathcal{L}[u] = pu^{p-1} + D_t^2 - D_x^2,$$

which is self-adjoint, $\mathcal{L}[u] = \mathcal{L}^*[u]$. Thus the adjoint system of the linearized field equation is given by

$$\mathcal{L}^*[u]\Lambda = pu^{p-1}\Lambda + D_t^2\Lambda - D_x^2\Lambda = 0, \tag{10}$$

where Λ is a scalar field which depends locally on u and derivatives of u, with u satisfying Eq. (9). System (10) is simply the determining equation for the local symmetries $\delta u = \Lambda[u]$ of the field equation (9). The solutions of the system are given by a time translation $\Lambda[u] = D_tu$, a space translation $\Lambda[u] = D_xu$, a Lorentz boost $\Lambda[u] = xD_tu + tD_xu$, and a scaling $\Lambda[u] = tD_tu + xD_xu + \alpha u$ with $\alpha = 2/(p-1) \neq 0$.

Through our adjoint invariance condition and conservation law formula, we now show that the time translation leads to the expected energy conservation law, while the scaling does not yield a conservation law. First, for $\Lambda[u] = D_tu$, we readily see

$$\mathcal{L}^*[U]\Lambda[U] = D_t(U^p + D_t^2U - D_x^2U) = D_tG[U]$$

for an arbitrary $U(x, t)$. Since $\partial\Lambda[U]/\partial U = 0$, $\partial\Lambda[U]/\partial D_tU = 1$, $\partial\Lambda[U]/\partial D_xU = 0$, etc., we have

$$-\frac{\partial\Lambda[U]}{\partial U}G[U] + D_t\left(\frac{\partial\Lambda[U]}{\partial D_tU}G[U]\right) + \cdots = D_t\left(\frac{\partial\Lambda[U]}{\partial D_tU}G[U]\right) = D_tG[U].$$

Thus $\Lambda[u] = D_tu$ satisfies the adjoint invariance condition (5). However, for $\Lambda[u] = tD_tu + xD_xu + \alpha u$, we find

$$\mathcal{L}^*[U]\Lambda[U] = (p\alpha + tD_t + xD_x)U^p + (2 + \alpha + tD_t + xD_x)(D_t^2U - D_x^2U) = (2+\alpha)G[U] + tD_tG[U] + xD_xG[U]$$

and

$$-\frac{\partial\Lambda[U]}{\partial U}G[U] + D_t\left(\frac{\partial\Lambda[U]}{\partial D_tU}G[U]\right) + D_x\left(\frac{\partial\Lambda[U]}{\partial D_xU}G[U]\right) + \cdots = (2-\alpha)G[U] + tD_tG[U] + xD_xG[U]$$

using $\partial\Lambda[U]/\partial U = \alpha$, $\partial\Lambda[U]/\partial D_t U = t$, $\partial\Lambda[U]/\partial D_x U = x$, etc. Since $\alpha \neq 0$, we see that the adjoint invariance condition (5) is not satisfied.

For any of the solutions $\Lambda[u]$ satisfying the adjoint invariance condition, the conservation law formula (7) leads to the conserved density

$$\Phi^0[u] = \int_0^1 d\lambda \Big(\Lambda[\lambda u] D_t u - u D_t \Lambda[\lambda u] + u \frac{1}{\lambda} \frac{\partial \Lambda[\lambda u]}{\partial D_t u} G[\lambda u] \Big). \quad (11)$$

Evaluating Eq. (11) for $\Lambda[u] = D_t u$ yields

$$\Phi^0[u] = \int_0^1 d\lambda (\lambda((D_t u)^2 - u D_x^2 u) + \lambda^p u^{p+1}) = \tfrac{1}{2}(D_t u)^2 + \tfrac{1}{2}(D_x u)^2 + \tfrac{1}{p+1} u^{p+1} - D_x(\tfrac{1}{2} u D_x u). \quad (12)$$

This is the expected energy density, up to a trivial conserved density.

Soliton equation.—Consider the Korteweg–de Vries (KdV) equation in physical form

$$G[u] = D_x^3 u + u D_x u + D_t u = 0. \quad (13)$$

This field equation lacks a direct Lagrangian formulation. Linearizing Eq. (13) leads to the operator

$$\mathcal{L}[u] = D_x^3 + D_x u + u D_x + D_t,$$

and taking the adjoint yields

$$\mathcal{L}^*[u] = -D_x^3 - u D_x - D_t \neq \mathcal{L}[u].$$

This leads to the adjoint system of the linearized KdV equation

$$\mathcal{L}^*[u]\Lambda = -D_x^3 \Lambda - u D_x \Lambda - D_t \Lambda = 0, \quad (14)$$

where Λ is a scalar field which depends locally on u and derivatives of u, with u satisfying Eq. (13).

Using our adjoint invariance condition and conservation law formula, we now *directly* derive the infinite sequence of local conservation laws [7] known for the KdV equation (13). By inspection, $\Lambda[u] = u$ is a solution of system (14) since $\mathcal{L}^*[u]u = -D_x^3 u - u D_x u - D_t u = -G[u] = 0$. An additional solution is easily found to be $\Lambda[u] = tu - x$ since $\mathcal{L}^*[u](tu - x) = -tD_x^3 u - tuD_x u - tD_t u - u + u = -tG[u] = 0$. There are no further solutions which are linear in u. Checking the adjoint invariance condition (5), for an arbitrary $U(x,t)$ we see that $-G[U]\partial\Lambda[U]/\partial U + \cdots$ reduces to $-G[U]$ if $\Lambda[U] = U$ and $-tG[U]$ if $\Lambda[U] = tU - x$. Since these expressions equal $\mathcal{L}^*[U]\Lambda[U]$, condition (5) is satisfied.

One can show that the recursion operator [8] $R^*[u] = D_x^2 + \frac{1}{3}u + \frac{1}{3}D_x^{-1}(uD_x)$ takes solutions of system (14) into solutions of system (14) since $\mathcal{L}^*[u]R^*[u] = R^*[u]\mathcal{L}^*[u]$. Consequently, the solutions $\Lambda[u] = tu - x$ and $\Lambda[u] = u$ each yield an infinite sequence of additional solutions. The solution sequence $\Lambda[u] = (R^*[u])^n(tu - x)$ for $n = 1, 2, \ldots$ has nonlocal dependence on u and thus fails to lead to local conservation laws. The other solution sequence $\Lambda[u] = (R^*[u])^n u$ for $n = 1, 2, \ldots$ depends locally on u and x derivatives of u. This sequence satisfies the adjoint invariance condition (5) and thus yields a sequence of local conservation laws.

From the conservation law formula (7) applied to the solutions $\Lambda[u]$ satisfying condition (5) we obtain the conserved density

$$\Phi^0[u] = \int_0^1 d\lambda (\Lambda[\lambda u] u). \quad (15)$$

Evaluating Eq. (15) for $\Lambda[u] = tu - x$ yields

$$\Phi^0[u] = \int_0^1 d\lambda (\lambda t u^2 - xu) = \tfrac{1}{2} t u^2 - xu. \quad (16)$$

For the sequence $\Lambda[u] = u$, $\Lambda[u] = R^*[u]u = D_x^2 u + \frac{1}{2}u^2$, $\Lambda[u] = (R^*[u])^2 u = D_x^4 u + \frac{5}{3}uD_x^2 u + \frac{5}{6}(D_x u)^2 + \frac{5}{18}u^3$, etc., we obtain

$$\Phi^0[u] = \int_0^1 d\lambda (\lambda u^2) = \tfrac{1}{2} u^2, \quad (17)$$

$$\Phi^0[u] = \int_0^1 d\lambda (\lambda u D_x^2 u + \lambda^2 \tfrac{1}{2} u^3) = \tfrac{1}{2} u D_x^2 u + \tfrac{1}{6} u^3, \quad (18)$$

$$\Phi^0[u] = \int_0^1 d\lambda (\lambda u D_x^4 u + \lambda^2 (\tfrac{5}{3} u^2 D_x^2 u + \tfrac{5}{6} u (D_x u)^2) + \lambda^3 \tfrac{5}{18} u^4), = \tfrac{1}{2} u D_x^4 u + \tfrac{5}{9} u^2 D_x^2 u + \tfrac{5}{18} u (D_x u)^2 + \tfrac{5}{72} u^4, \quad (19)$$

etc. Equation (16) and Eqs. (17)–(19), etc., agree with the local conserved densities found in Ref. [7], up to addition of trivial conserved densities.

Heat equation.—Consider the equation for heat conduction

$$G[u] = D_x^2 u - D_t u = 0. \quad (20)$$

This is a dissipative field equation inherently lacking a Lagrangian formulation. Through our adjoint invariance condition and conservation law formula, we now derive the elementary local conservation laws of Eq. (20).

Since Eq. (20) is linear, we directly obtain the adjoint system

$$\mathcal{L}^*[u]\Lambda = D_x^2 \Lambda + D_t \Lambda = 0, \quad (21)$$

where Λ is a scalar field. System (21) has solutions $\Lambda = f(x,t)$ given by

$$D_x^2 f + D_t f = 0 \quad (22)$$

with no dependence on u. These solutions trivially satisfy the adjoint invariance condition (5) since $\mathcal{L}^*[U]\Lambda$ and $\partial\Lambda/\partial U$, $\partial\Lambda/\partial D_t U$, $\partial\Lambda/\partial D_x U$, etc. identically vanish for an arbitrary $U(x,t)$. There are no solutions which have explicit local dependence on u or derivatives of u.

Evaluating the conservation law formula (7) for $\Lambda = f(x,t)$, we obtain the conserved density

$$\Phi^0[u] = \int_0^1 d\lambda(\Lambda u) = uf\,, \tag{23}$$

where $f(x,t)$ is an arbitrary solution of the adjoint heat equation (22). This yields the infinite number of elementary local conserved densities of the heat equation (20).

Nonlinear diffusion equation.—Consider Burgers' equation

$$G[u] = D_x^2 u - uD_x u - D_t u = 0\,, \tag{24}$$

which is a nonlinear dissipative field equation with no Lagrangian formulation. Burgers' equation is related to the heat equation by a *nonlocal* transformation [9] involving solutions of the adjoint system of the linearization of Eq. (24). Using these solutions and our conservation law formula, we now derive corresponding conservation laws of Burgers' equation.

From the divergence form of Eq. (24) we introduce a potential $v(x,t)$ such that

$$D_x v = u, \qquad D_t v = D_x u - \tfrac{1}{2}u^2. \tag{25}$$

The field equation for v is given by

$$G[v] = -D_t v + D_x^2 v - \tfrac{1}{2}(D_x v)^2 = 0\,. \tag{26}$$

Linearizing Eq. (26) and taking the adjoint leads to the adjoint system

$$\mathcal{L}^*[v]\Lambda = D_t\Lambda + D_x^2\Lambda + D_x(\Lambda D_x v) = 0\,, \tag{27}$$

where Λ is a scalar field which depends locally on v and derivatives of v, with v satisfying Eq. (26). System (27) is equivalent to the adjoint system (21) of the heat equation through the local transformation $\Lambda \to \Lambda e^{-v/2}$. Thus, $\Lambda[v] = e^{-v/2}f(x,t)$ yields solutions of system (27), where $f(x,t)$ satisfies the adjoint heat equation (22).

It is straightforward to check $\Lambda[v]$ satisfies $\mathcal{L}^*[V]\Lambda[V] = -G[V]\partial\Lambda[V]/\partial V$ for an arbitrary $V(x,t)$, and thus the adjoint invariance condition (5) holds. Consequently, from our conservation law formula (7), we obtain the conserved density

$$\begin{aligned}\Phi^0[v] &= \int_0^1 d\lambda(\Lambda[\lambda v]v) \\ &= \int_0^1 d\lambda e^{-\lambda v/2} vf = 2f - 2e^{-v/2}f\,.\end{aligned} \tag{28}$$

The term $2f$ in Eq. (28) is separately conserved due to f satisfying the adjoint heat equation (22). Since f is an arbitrary solution, the remaining term $-2e^{-v/2}f$ yields an infinite number of conserved densities for system (25). Eliminating v in terms of u, which we take to have compact support in x, we obtain the following conserved densities for Burgers' equation (24):

$$\Phi^0[u] = e^{-D_x^{-1}u/2}f\,, \tag{29}$$

where D_x^{-1} represents an antiderivative with respect to x.

In addition to our algorithm, we can also find conservation laws directly for any given system of field equations (1) through the identity

$$W^\Omega \mathcal{L}_{\Omega\sigma}[U]V^\sigma - V^\sigma \mathcal{L}^*_{\sigma\Omega}[U]W^\Omega = D_i S^i[W,V]\,, \tag{30}$$

which holds for arbitrary V^σ, W^Ω, where

$$\begin{aligned}S^i[W,V] = {} & W^\Omega G^i_{\Omega\sigma}[U]V^\sigma \\ & + (W^\Omega G^{ij}_{\Omega\sigma}[U]D_j V^\sigma \\ & - D_j(W^\Omega G^{ij}_{\Omega\sigma}[U])V^\sigma) + \cdots\,.\end{aligned} \tag{31}$$

Given $\eta^\sigma[u]$ satisfying the linearized system (2) and $\Lambda^\Omega[u]$ satisfying the adjoint system (4), we see by Eqs. (30) and (31) that

$$\begin{aligned}\Phi^i[u] = {} & \Lambda^\Omega[u]G^i_{\Omega\sigma}[u]\eta^\sigma[u] \\ & + (\Lambda^\Omega[u]G^{ij}_{\Omega\sigma}[u]D_j\eta^\sigma[u] \\ & - D_j(\Lambda^\Omega[u]G^{ij}_{\Omega\sigma}[u])\eta^\sigma[u]) + \cdots\end{aligned} \tag{32}$$

yields a conservation law (6) on all solutions u^σ of the field equations (1). Here $\eta^\sigma[u]$ and $\Lambda^\Omega[u]$ are allowed to have other than just a local dependence on u^σ. This extends corresponding results obtained for self-adjoint systems of field equations in Ref. [10].

In a forthcoming paper [6] we show that the adjoint invariance condition (5) and the conservation law formula (7) can be generalized to hold when $\Lambda^\Omega[u]$ has nonlocal dependence on u^σ. We further show that, conversely, any conservation law having nonlocal dependence on u^σ arises from some such nonlocal $\Lambda^\Omega[u]$ satisfying the adjoint system (4). As a consequence, it follows that for any given system of field equations, all conservation laws, local and nonlocal, arise from finding all solutions $\Lambda^\Omega[u]$ of the adjoint system (4).

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[4] Greek indices represent any internal indices together with any vector, tensor, and spinor indices of the fields and field equations. Latin indices represent time and space coordinate indices. Summation is assumed over any repeated index in all expressions.
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