Spatial Modulational Instability and Multisolitonlike Generation in a Quadratically Nonlinear Optical Medium

R. A. Fuerst, D.-M. Baboiu, B. Lawrence, W. E. Torruellas,* and G. I. Stegeman
Center for Research and Education in Optics and Lasers and Department of Physics, University of Central Florida, Orlando, Florida 32826

Stefano Trillo and Stefan Wabnitz†
Fondazione Ugo Bordoni, Via B. Castiglione 59, 00142 Roma, Italy
(Received 17 October 1996)

A novel type of spatial modulational instability induced by the *dynamical* interaction of two strongly coupled fundamental and harmonic fields in a second-order nonlinear optical material is demonstrated experimentally. This phenomenon is explained theoretically on the basis of a one-dimensional Floquet theory. At high intensities, the formation of a 1D solitary wave lattice is superseded by the onset of 2D modulational instabilities. [S0031-9007(97)02848-2]

PACS numbers: 42.65.Tg, 42.65.Ky, 42.65.Sf

Instabilities and chaos can occur in a wide class of wave phenomena when intense waves propagate through nonlinear media [1]. Among several examples investigated for electromagnetic waves, the self-filamentation of single beams, due to the self-focusing effects inherent to positive third order nonlinearities, was observed in the early days of nonlinear optics [2]. A one dimensional (1D) temporal counterpart of this phenomenon is the breakup of a cw field in an optical fiber [3] due to modulational instability (MI). In Kerr media, the propagation is described by the nonlinear Schrödinger equation (NLSE) whose integrability ensures the existence of extremely stable solutions in 1D, i.e., in slab waveguide geometries. Conversely, in 2D blow-up instabilities occur unless higher order effects becomes significant [4,5]. Infinite beams such as plane waves [6] or soliton stripes [7] experience MI in 2D. These instabilities are also important in water waves [6] and plasmas [8]. The dimensionality also affects features in the long range evolution of MI: exact recurrence (periodic reconstruction of the plane wave) occurs in the 1D NLSE [9], whereas pseudorecurrence takes place in 2D [6,10]. In the presence of dissipative perturbations such as driving and damping in a Kerr cavity [11], or an adiabatic variation of the dispersion in a fiber [12], these instabilities lead to the formation of periodic structures which propagate without spreading in space or time.

Recently spatial solitons consisting of two strongly coupled optical fields have been observed in the second harmonic generation (SHG) process [13]. Theory has shown that these solitons are stable [14]. Because of their robustness, mutual trapping of the fundamental (FW) and SH beams occurs, even when the latter is not launched. The typical size of 2D solitons requires strong focusing. Conversely, in the quasiplane-wave regime, this strong coupling between two fields at different frequencies introduces new degrees of freedom into MI phenomena. MI was recently predicted to occur for the plane-wave nonlinear eigenmodes of SHG [15,16]. These correspond to special launching conditions requiring injection of both FW and SH plane waves with a given relative phase that propagate without exchanging energy and remaining phase locked [17]. In this Letter we show, however, that 1D spatial MI builds up under more general conditions, namely, in SHG from FW input beams (with highly elliptical cross section). To our knowledge, this is the first observation of a *dynamical* transverse instability, i.e., MI occurring from an intense beam composed by two strongly interacting frequency modes. We develop a theoretical approach which exposes new features of MI.

The propagation equations for SHG are well known. We make use of type-II SHG (i.e., the same geometry of Ref. [13]) involving two orthogonally polarized FW inputs. Since highly elliptical beams (quasiplane wave in the transverse coordinate x) are employed, the propagation is reasonably described by the 1D model

$$\begin{aligned} i\frac{\partial a_1}{\partial Z} - \frac{r_1}{2}\frac{\partial^2 a_1}{\partial X^2} + a_3 a_2^* e^{-i\beta Z} &= 0\,, \\ i\frac{\partial a_2}{\partial Z} - i\delta_2\frac{\partial a_2}{\partial X} - \frac{r_2}{2}\frac{\partial^2 a_2}{\partial X^2} + a_3 a_1^* e^{-i\beta Z} &= 0\,, \\ i\frac{\partial a_3}{\partial Z} - i\delta_3\frac{\partial a_3}{\partial X} - \frac{r_3}{2}\frac{\partial^2 a_3}{\partial X^2} + a_1 a_2 e^{i\beta Z} &= 0\,, \end{aligned} \tag{1}$$

where a_1 (ordinary polarized FW), a_2 (extraordinary polarized FW), and a_3 (SH) are envelope amplitudes of the three waves. We use the following normalization: Z is the distance in units of the parametric length $z_{\rm nl} = (\chi\sqrt{I_{\rm tot}})^{-1}$, $X = x/x_{tr}$ where $x_{tr} = \sqrt{z_{\rm nl}/k_1}$, $I_{\rm tot}$ is a (plane-wave) total intensity, and $\beta = (k_1 + k_2 - k_3)z_{\rm nl} = \Delta kL(z_{\rm nl}/L)$ is the mismatch parameter, L being the sample length. In KTP, $d_{\rm eff}^{(2)} = 2.8$ pm/V resulting in a nonlinear coefficient $\chi \approx 5 \times 10^{-4}$ W$^{-1/2}$. For the spatial case, $r_1 = -1$, $r_2 = -k_1/k_2 \approx -1$, and $r_3 = -k_1/k_3 \approx -1/2$, since all of the refractive indices are equal to within a few percent. We found that the essential physics is not affected by the spatial walk-off terms, and hence we set $\delta_2 = \delta_3 = 0$. Under these conditions, whenever the two FW beams are equally excited, the first two of Eqs. (1) become degenerate, and are equivalent to scalar type-I SHG. The equations governing the latter interaction are formally obtained by the substitution $a_1 = a_2 = a_0/\sqrt{2}$ in Eqs. (1). When a plane FW (i.e., X independent) is launched as an initial condition, it is well known that Eqs. (1) predict that the SH is generated and reconverted back in a periodic fashion [18]. As the following analysis shows, MI is still expected to build up over a distance of the order of the period of the plane-wave interaction. The stability calculation is carried out by inserting in Eqs. (1) the ansatz

$$a_0(Z,X) = [\rho_0(Z) + \epsilon_0(Z,X)]\exp[i\phi_0(Z)], \quad (2)$$

$$a_3(Z,X) = [\rho_3(Z) + \epsilon_3(Z,X)]\exp[i\phi_3(Z) + i\beta Z], \quad (3)$$

where the two (FW and SH) perturbations $\epsilon_{0,3}$ read as sideband pairs in k_x space

$$\epsilon_j(Z,X) = \epsilon_{js}(Z)e^{iK_xX} + \epsilon_{ja}(Z)e^{-iK_xX}, \qquad j = 0,3\,. \quad (4)$$

Here the novelty with respect to conventional MI [15] is due to the Z dependence of the plane waves $a_j = \rho_j\exp(i\phi_j)$. We linearize in ϵ, and group terms with the same spatial frequency. The plane-wave ($k_x = 0$) contributions obey the system $\dot\eta_3 = -\dot\eta_0/2 = \eta_0\sqrt{\eta_3}\sin\phi = \partial H_r/\partial\phi$, $\dot\phi = [(1 - 3\eta_3)/\sqrt{\eta_3}]\cos\phi - \beta = -\partial H_r/\partial\eta_3$, where $\eta_3 = \rho_3^2$, $\eta_0 = \rho_0^2 = 2(1 - \eta_3)$, $\phi = \phi_3 - 2\phi_0$, and $H_r = 2\sqrt{\eta_3}(1 - \eta_3)\cos\phi - \beta\eta_3$ is the Hamiltonian [17]. The contributions at $k_x = \pm K_x$ yield the linear system for the perturbation $\epsilon \equiv (\epsilon_{0s}, \epsilon_{0a}^*, \epsilon_{3s}, \epsilon_{3a}^*)^T$

$$\dot\epsilon = M(Z)\epsilon\,, \quad (5)$$

where the 4×4 matrix $M = \{m_{ij}\}$ has the diagonal elements $m_{11} = -m_{22} = r_1K_x^2/2 - \rho_3\cos\phi$, $m_{33} = -m_{44} = r_3K_x^2/2 - (\rho_0^2/2\rho_3)\cos\phi$, and the nonvanishing off-diagonal elements $m_{12} = -m_{21}^* = \rho_3\exp(i\phi)$, and $m_{13} = m_{31}^* = -m_{24}^* = -m_{42} = \rho_0\exp(i\phi)$. Hence the perturbation evolves according to the linear system (5) with periodic coefficients, the periodicity being introduced through the plane-wave solutions. For instance, our experiment of nonseeded SHG is described by the periodic plane-wave solution $\eta_3(Z) = \eta^-\mathrm{sn}^2(\sqrt{\eta^+}Z|k)$, with the period $Z_p = 2K(k)/\sqrt{\eta^+}$, where $k^2 = \eta^-/\eta^+$ is the modulus of the elliptic integral of the first kind $K(k)$, and $\eta^\pm = 1 + \beta^2/8 \pm (\beta/4)[(\beta/4)^2 + 1]^{1/2}$ [18]. By applying Floquet theory to Eqs. (5), the stability depends on the so-called Floquet multipliers or characteristic exponents [19]. These are obtained by constructing the 4×4 fundamental matrix solution U, whose columns are the solutions to Eqs. (5) at $Z = Z_p$ for the four initial values $\epsilon(0) = (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. MI occurs when one of the eigenvalues λ of U is $|\lambda| > 1$. In this case the (intensity) growth rate of the instability is given by the characteristic exponent $g = 2\ln|\lambda|/Z_p$. We computed the MI gain as a function of K_x for different values of β. In Fig. 1, we show the physical gain $G = gz_{\mathrm{nl}}^{-1}$ for our KTP sample of $L = 1$ cm versus the spatial frequency $\nu_x = K_x/(2\pi x_{tr})$. Figure 1(a) shows the gain close to phase matching, i.e., for $\Delta kL = 10^{-2}$, and intensities $I_{\mathrm{tot}} = 10, 50, 100$ GW/cm^2 corresponding to $\beta \simeq 20, 9, 6 \times 10^{-5}$. Figure 1(b) shows how the expected spectral gain changes when moving off phase matching (e.g., $\Delta kL = \pm 3\pi$, corresponding to $\beta \simeq \pm 0.2$), for a fixed intensity $I_{\mathrm{tot}} = 10$ GW/cm^2.

The fluctuations lead the most unstable perturbations to spontaneously break the plane wave into a periodic sequence of peaks. The nonintegrability of the SHG Eqs. (1) suggests that MI on the infinite line will evolve into a turbulent state with generation of multiple frequencies [15]. Conversely, an initial beam with a finite width, will split into a number of beams roughly given by the ratio of the beam width to the period of the most unstable modulation. For a Gaussian beam, the density of spots is expected to decrease in the tails. We believe that the beam envelope has a stabilizing effect on the formation of the multipeak structure. In order to support this argument, we numerically propagated a wide, 1D, FW Gaussian beam ($a_1 = a_2 = \exp[-(X/20)^2]$), perturbed by white amplitude noise with a Gaussian distribution (no SH at the input, $\beta = 2$). To prove that beam breakup is indeed due to noise-driven MI, Fig. 2(a) shows the evolution of a wide FW Gaussian in the absence of noise. The FW envelopes only exhibit the presence of oscillations due to the different SHG experienced by the portions of the envelope which see different parametric lengths [$z_{\mathrm{nl}} \propto I_{\mathrm{tot}}^{-1/2}(X)$]. The presence of the noise, however, dramatically changes the results. Figure 2(b) shows the evolution of the same beam as in Fig. 2(a), but with amplitude fluctuations averaging 10%, typical of our experiment. During propagation those components with the most unstable transverse periodicity ($2\pi/K_x \approx 6$ from our theory) dominate. Here MI induced by the interplay of diffraction and SHG leads

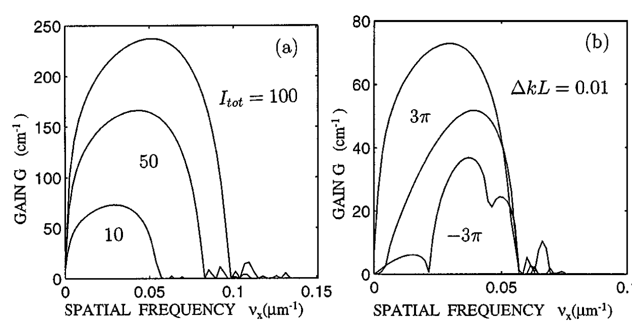

FIG. 1. MI gain G versus spatial frequency ν_x. (a) Fixed $\Delta kL = 10^{-2}$ and different intensities. (b) Fixed intensity $I_{\mathrm{tot}} = 10$ GW/cm^2 and different values of $\Delta kL = 0.01, \pm 3\pi$.

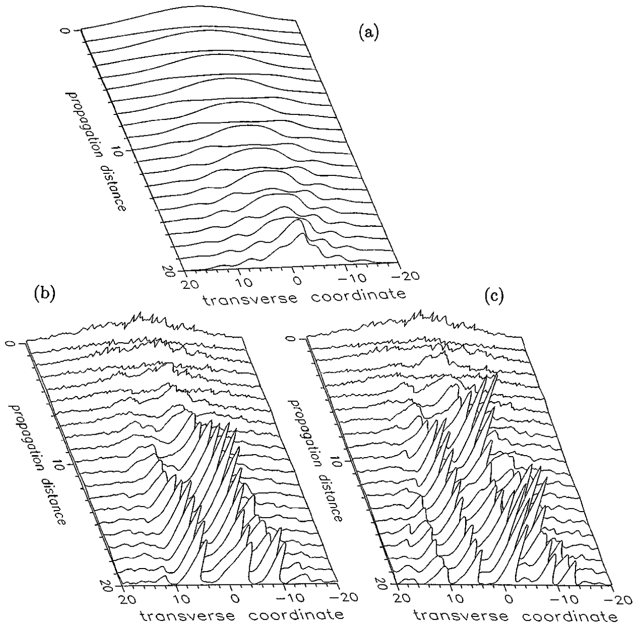

FIG. 2. Evolution of a wide FW Gaussian beam close to phase matching of SHG (a) in the absence of noise; (b) same as (a) with 10% amplitude noise superimposed on the input; (c) same as (b) but with twice the input power.

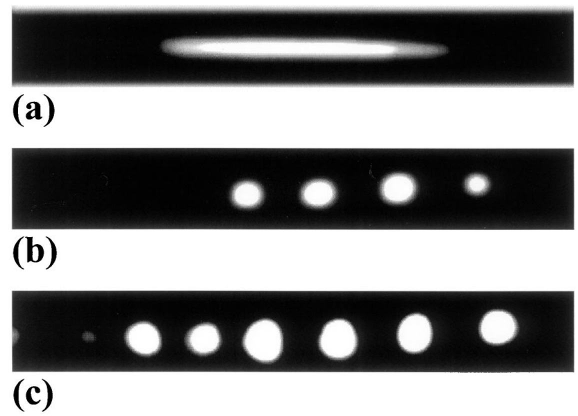

FIG. 3. (a) The input beam. (b) The output at 48 $\mathrm{GW/cm^2}$. (c) The output at 57 $\mathrm{GW/cm^2}$.

to the noise-driven breakup of both the FW and the generated SH (not shown) beams. In this case the original beam splits into three main separated beams, propagating in parallel directions. Doubling the intensity of the input beam, the output number of maxima increase [see Fig. 2(c)]. The resulting structure is not completely stable, even though the beam sizes and powers are typical of quadratic solitons. The beams experience direct interaction forces and exchange of radiation as well. In 2D we expect both qualitative and quantitative differences: the radiation can leave the interaction region along the orthogonal coordinate Y, thereby (i) favoring the stabilization into quadratic solitons, and (ii) reducing the fraction of input power converted into the MI pattern.

The MI was experimentally demonstrated in a $L = 1$ cm long KTP crystal (see Ref. [13] for the details). The source used was a Q-switched, mode-locked, Nd:YAG providing 35 psec pulses with 10 Hz repetition, at $\lambda_0 = 1064$ nm. The incident polarization was held fixed at 45° between the axes, providing a total measured peak intensity $I_{\mathrm{tot}} = 2I_o = 2I_e$. A variable elliptical beam was created with an adjustable cylindrical telescope. The small dimension of the incident beam was maintained at a FWHM value of 20 μm. The larger dimension was varied such that aspect ratios (X to Y) above 5:1 were created at the input face of the crystal. This makes a 1D plane-wave approximation reasonable since in a 1 cm long crystal, the effects of diffraction on the spatial envelopes are minimal for beams with such large waists. Figure 3 illustrates qualitatively the dependence of the spatial frequency of the MI on the incident intensity for an elliptical beam with an aspect ratio of 12.0:1. Figure 3(a) shows the profile of the incident FW beam at the focus which is located at the front face of the KTP crystal, while 3(b) and 3(c) show the FW beam profiles at the exit face of the crystal for intensities of 48 and 57 $\mathrm{GW/cm^2}$, respectively. The breakup of the beam into a line of well-defined circular spots is clear. Cuts through the spots in the orthogonal directions showed them to be circular to within $\pm 8\%$ in the central part of the pattern. Their radius was 9.5 ± 1.5 μm, essentially equal to the value of 10–12 μm, obtained for spatial solitons when 20 μm circular beams are incident onto this crystal [13]. The patterns shown in Fig. 3 were obtained with single laser shots, while the noise was estimated by recording successive shots. The absolute location (but not the shape or separation) of the beams jittered in space along the long ellipse axis from shot to shot, showing the noise generated aspects of the patterns. Although not shown clearly in the high contrast photographs, weak FW fields were also observed in some cases between the spots, corresponding to the expected radiation exchange between the solitonlike beams. Furthermore, as the input intensity is increased, there was no indication of any return to the plane-wave elliptically shaped field, as predicted. These results strongly support our conjecture that the beams are indeed spatial solitons.

The beam breakup was also investigated for two phase-mismatch parameters $\Delta kL = \pm 3\pi$. The most detailed set of data was collected at $\Delta kL = -3\pi$ because this provides a larger intensity increment between increases in the number of beams generated. Although rigorously MI has no threshold intensity, its observability requires a minimum intensity for the exponential gain to become significant over the length L. For instance, a gain factor exp(20) requires $gL/z_{\mathrm{nl}} = 20$ which yields a cw plane-wave threshold $I_{\mathrm{th}} \approx 3$ $\mathrm{GW/cm^2}$ at $\Delta kL = -3\pi$, in sufficient agreement with the observation, taking into account the measured radiation losses, and the temporal structure of the beams. We also observe a reduction of the threshold intensity I_{th} for the positive β, the lowest threshold being at phase matching, in qualitative agreement with the theoretical prediction [see Fig. 1(b)]. However, it is the *dependence of the breakup frequency on the intensity* which constitutes the most clear signature of spontaneous MI [15]. This feature is clearly shown in Fig. 4, where we report the number of spots N_s in the observed stable patterns versus intensity for two ellipticities. The intensity required to see a given N_s increases for the smaller ellipticity. The data with ellipticity 10.8:1 are closer to the plane-wave 1D case, for which the MI theory predicts $N_s \sim \nu_x \propto I_{\mathrm{tot}}^{1/4}$ for a large N_s [due to the scaling law $x_{tr} = (k_1 \chi \sqrt{I_{\mathrm{tot}}})^{-1/2}$ of Eqs. (2)]. However, we do not expect a detailed fit for the current experimental parameters which yield a relatively small N_s.

For 2D waves the validity of our 1D model is expected to hold best when the profile of the beam across the other transverse coordinate y is strongly confined and constant. This happens strictly in waveguides, but also for a strip beam in a bulk medium, where the growth rate of the transverse instability depends smoothly on the preserved profile in the other dimension [20]. Conversely, when the profile across y exhibits large changes, the 1D theory is not expected to work. In our experiment, at very high input intensities $I_{\mathrm{tot}} \sim 150$ GW/cm^2 for an elliptical beam with a large aspect ratio of 12.0:1, we observed the onset of beam breakup along the short ellipse axis. It occurred near the center of the input beam at intensities for which the soliton spacing along x was comparable to the y beam dimension. This is indicative of 2D pattern formation, and constitutes experimental evidence for the breakdown of the 1D picture. However, damage limitations precluded a detailed investigation of this situation.

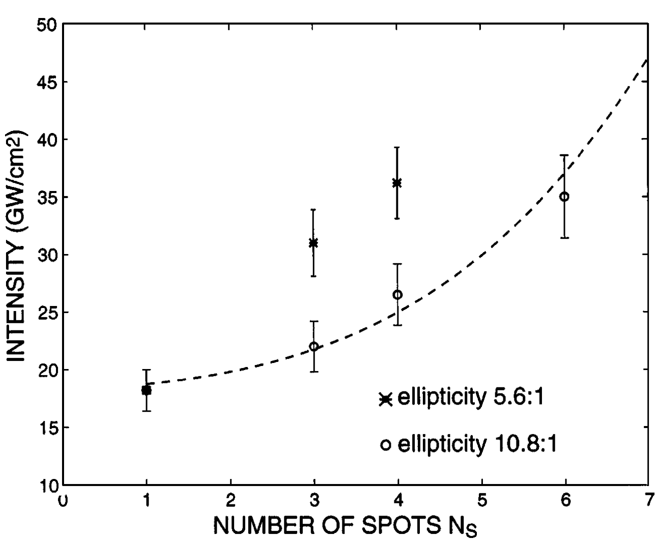

FIG. 4. Number of separate beams observed at the output versus the intensity for two different ellipticities of the input beam, and $\Delta kL = -3\pi$. The dashed line is a best fit with a quartic polynomial in N_s for the data with ellipticity 10.8:1.

In summary, spatial MIs have been investigated both experimentally and theoretically in quadratically nonlinear media near phase-matched SHG. Experimentally, highly elliptical input beams were observed to break up into a line of circular, separated beams whose measured properties strongly suggested them to be quadratic spatial solitons. At very high input intensities, the onset of a 2D pattern was observed. We have presented a 1D theory which accounts for the dynamical behavior of the beams and agreed well with the experimental results.

This research was supported by DARPA, ARO, and NSF.

*Permanent address: Department of Physics, Washington State University, Pullman, WA 99164.

†Permanent address: Physics Lab., Bourgogne University, 21004 Dijon, France.

[1] E. Infeld and R. Rowlands, *Nonlinear Waves, Solitons and Chaos* (Cambridge Press, Cambridge, 1990).
[2] S. A. Akhmanov *et al.*, in *Laser Handbook*, edited by F. T. Arecchi and E. O. Schulz-DuBois (North-Holland, Amsterdam, 1972), pp. 1151–1228.
[3] K. Tai, A. Hasegawa, and A. Tomita, Phys. Rev. Lett. **56**, 135 (1986).
[4] For example, A. C. Newell, *Solitons in Mathematics and Physics* (Soc. Ind. Appl. Math., Philadelphia, 1985).
[5] P. L. Kelley, Phys. Rev. Lett. **15**, 1005 (1965).
[6] H. C. Yuen and W. E. Ferguson, Phys. Fluids **21**, 2116 (1978); D. U. Martin and H. C. Yuen, Phys. Fluids **23**, 881 (1980).
[7] V. E. Zakharov and A. M. Rubenchik, Sov. Phys. JETP **38**, 494 (1973); A. V. Mamaev, M. Saffman, and A. A. Zozulya, Europhys. Lett. **35**, 25 (1996).
[8] P. Sprangle *et al.*, Phys. Rev. Lett. **73**, 3544 (1994).
[9] E. Infeld, Phys. Rev. Lett. **47**, 717 (1981); M. J. Ablowitz and B. M. Herbst, Phys. Rev. Lett. **62**, 2065 (1989).
[10] N. N. Akhmediev *et al.*, Phys. Rev. Lett. **65**, 1423 (1990).
[11] L. A. Lugiato and R. Lefever, Phys. Rev. Lett. **58**, 2209 (1987); M. Haelterman, S. Trillo, and S. Wabnitz, Phys. Rev. A **47**, 2344 (1993).
[12] E. M. Dianov *et al.*, Opt. Lett. **14**, 1008 (1989).
[13] W. E. Torruellas *et al.*, Phys. Rev. Lett. **74**, 5036 (1995).
[14] V. E. Zakharov, S. L. Musher, and A. M. Rubenchik, Phys. Rep. **129**, 285 (1985); D. E. Pelinovsky, A. V. Buryak, and Y. S. Kivshar, Phys. Rev. Lett. **75**, 591 (1995); S. K. Turitsyn, JETP Lett. **61**, 469 (1995).
[15] S. Trillo and P. Ferro, Opt. Lett. **20**, 438 (1995); Phys. Rev. E **51**, 4994 (1995).
[16] A. V. Buryak and Yu. S. Kivshar, Opt. Lett. **19**, 1612 (1994); Phys. Rev. A **51**, R41 (1995).
[17] S. Trillo *et al.*, Opt. Lett. **17**, 637 (1992); A. E. Kaplan, Opt. Lett. **18**, 1223 (1993).
[18] J. A. Armstrong *et al.*, Phys. Rev. **6**, 1918 (1962).
[19] E. A. Coddington and N. Levison, *Theory of Ordinary Differential Equations* (McGraw-Hill, New York, 1955).
[20] N. N. Akhmediev, V. I. Korneev, and R. F. Nabiev, Opt. Lett. **17**, 393 (1992).