Systems with Multiplicative Noise: Critical Behavior from KPZ Equation and Numerics

Yuhai Tu, G. Grinstein, and M. A. Muñoz
IBM Research Division, T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598
(Received 13 August 1996)

We show that certain critical exponents of systems with multiplicative noise can be obtained from exponents of the KPZ equation. Numerical simulations in 1D confirm this prediction and yield other exponents of the multiplicative noise problem. The numerics also verify an earlier prediction of the divergence of the susceptibility over an entire range of control parameter values and show that the exponent governing the divergence in this range varies continuously with control parameter. [S0031-9007(96)02067-4]

PACS numbers: 64.60.Ht, 02.50.–r, 47.20.Ky

Langevin equations—first order partial differential equations in time containing Gaussian random noise terms—capture the macroscopic physics of many classical, stochastic, many-body systems [1]. In the most common situation, one that includes much of equilibrium statistical mechanics, the noise amplitude is simply a constant. There are, however, important classes of problems in which the noise amplitude is proportional to a positive power α of the field variable itself. The well-known case $\alpha = \frac{1}{2}$ describes the physics of directed percolation and its many and diverse realizations [2].

This paper deals with "multiplicative-noise" systems (MNS) [3,4], wherein the dominant source of noise is external, and hence $\alpha = 1$. As an example, consider the chemical reactions $A + X \underset{k_2}{\overset{k_1}{\rightleftharpoons}} 2X$, $B + X \xrightarrow{k_3} C$, for chemical species A, B, C, X, and rate constants k_1, k_2, k_3. The phenomenological reaction-diffusion equation for the coarse-grained density $n(\vec{x}, t)$ of X particles is [3] $\partial n/\partial t = \nabla^2 n + \gamma_1 n_A n - \gamma_2 n^2 - \gamma_3 n_B n$, where the constants $\gamma_i \propto k_i$ for $i = 1, 2, 3$. Suppose one tries experimentally to keep the densities $n_{A,B}$ fixed externally at values $n^0_{A,B}$. Despite one's best efforts there will be local fluctuations $\delta n^0_{A,B}(\vec{x}, t)$. Substituting $n_{A,B}(\vec{x}, t) = n^0_{A,B} + \delta n^0_{A,B}(\vec{x}, t)$ into the above equation yields the generic MNS with a single-component field

$$\partial n(\vec{x}, t)/\partial t = \nabla^2 n - rn - un^{2+\rho} + n\eta\,. \qquad (1)$$

Here $r \equiv \gamma_1 n^0_A - \gamma_3 n^0_B$, $u \equiv \gamma_2$, and the noise $\eta \equiv \gamma_1 \delta n_A - \gamma_3 \delta n_B$ is, without loss of generality, taken to be Gaussian with strength D: $\langle \eta(\vec{x}, t)\eta(\vec{x}', t')\rangle = D\delta(\vec{x} - \vec{x}')\delta(t - t')$. Though the chemical reaction problem corresponds to $\rho = 0$, we also consider $\rho = 1$.

In a previous paper [5] we analyzed the phase structure of this model, which has two phases, an "active" phase with $\langle n\rangle > 0$ and an "absorbing" phase with $n(\vec{x}) = 0$ for all $\vec{x}$, that occurs for sufficiently large r. These phases are separated by a critical point at $r \equiv r_c$, where $r_c = 0$ in mean-field theory. [Note that the vanishing of the noise amplitude with n makes the right side of Eq. (1) vanish when $n(\vec{x})$ vanishes, thereby causing all dynamics to cease and making the absorbing configuration $n(\vec{x}) = 0$ a potentially stable phase.] We showed that model (1) had a critical dimension $d_c = 2$, below which the transition is governed by an analytically inaccessible strong-coupling fixed point. For $d > 2$, on the other hand, the transition is governed by the "weak-coupling" Gaussian fixed point with mean-field exponents for D less than a critical value D_c. For $D > D_c$, however, the strong-coupling fixed point is the stable one. A multicritical point occurs at $D = D_c$.

In this paper we argue that the critical behavior of MNS is actually governed by the fixed point of the Kardar-Parisi-Zhang (KPZ) model of growing interfaces [6]. This mapping is consistent with the phase diagram proposed in Ref. [5], and with known exponents of the weak-coupling and multicritical fixed points. It also allows us to express the dynamical exponent z and the correlation length exponent ν of the strong-coupling transition in terms of the KPZ exponents. These exponents are found to be independent of the degree $2 + \rho$ of the nonlinearity. A lower bound of unity for the order parameter exponent β in the case $\rho = 0$ also emerges. We confirm these predictions by calculating numerically in 1D the four independent exponents characterizing the strong-coupling transition. We also confirm the rather striking prediction in Ref. [5] of an entire domain of r values, encompassing the critical value r_c, in which the susceptibility of the system diverges. The numerics show that, as in the exactly solvable single variable (0D) problem, this region of infinite susceptibility extends to both sides of the critical point and is controlled by a fixed line with continuously varying critical exponents [5].

To establish the connection between model (1) and the KPZ theory, note that the field $n(\vec{x}, t)$ in (1) will remain positive if $n(\vec{x}, 0) > 0$ for all x. In this case one can perform the Hopf-Cole [7] change of variable $n(\vec{x}, t) = e^{h(\vec{x},t)}$, producing the equation [8]

$$\partial h/\partial t = -r + \nabla^2 h + (\nabla h)^2 - ue^{(1+\rho)h} + \eta\,. \qquad (2)$$

Aside from the u term this is precisely the KPZ equation [wherein the standard KPZ nonlinearity $(\nabla h)^2$ has coefficient unity]. Note, however, that either in the absorbing phase or at the critical point, the steady-state value of n is zero, whereupon the steady-state value of h is $-\infty$. Thus the u term vanishes in steady state, leaving one with precisely the KPZ theory.

 0031-9007/97/78(2)/274(4)$10.00

Recall [6] that the phase diagram of the KPZ equation consists of a unique strong coupling phase for $d \leq d_c = 2$, and both weak and strong coupling phases separated by a multicritical point for $d > 2$. The weak and strong-coupling regimes occur for noise strengths D that are, respectively, smaller and larger than a critical value. This phase diagram is thus encouragingly similar to that of the MNS. Also the dynamical critical exponent z is 2 for both MNS [5] and KPZ [9] along the line of weak-coupling transitions and at the multicritical point, for $d > 2$.

Let us now consider the critical exponents for the strong-coupling transition. The dynamical critical exponent z for the MNS can be computed from the steady-state behavior of response functions right at the critical point, and hence is identical to the value of z in the KPZ theory. In particular, then, $z = \frac{3}{2}$ for $d = 1$ [6].

To argue that other exponents of the MNS can also be obtained from the KPZ equation requires us to consider the active state of model (1). For r slightly less than r_c, $0 < \langle n \rangle \ll 1$ in steady state, implying that $h_0 \equiv \langle h \rangle$ is very large and negative. Writing $h(\vec{x}, t) \equiv \langle h \rangle + \delta h(\vec{x}, t)$, one obtains an equation for δh that is identical to the KPZ equation except for the extra nonlinear term $-u' e^{(1+\rho)\delta h}$, where $u' \equiv u e^{(1+\rho)h_0}$. Noting that the leading nontrivial term in the expansion of this nonlinearity in powers of δh is the linear "mass" term $-u'(1 + \rho)\delta h$, we infer that the main effect of $-u' e^{(1+\rho)\delta h}$ is to produce a finite correlation length ξ at which the power law correlations of the KPZ equation are cut off and replaced by exponential behavior. One concludes that ξ must be the correlation length of the corresponding MNS; its divergence as $r \rightarrow r_c$ is governed by the critical exponent ν. In the critical region, i.e., on length scales $|\vec{x}| \ll \xi$, the u' term is negligible, so critical correlations can be computed from the KPZ equation.

To calculate ν from KPZ, take the expectation value of the δh equation, recall that $\langle \delta h \rangle = 0$, and write the extra nonlinear term as $-u\langle n^{1+\rho} \rangle$, to obtain

$$-r + \langle (\nabla h)^2 \rangle - u\langle n^{1+\rho} \rangle = 0 . \tag{3}$$

At the critical point, $r = r_c$ and $n = 0$, so $-r_c + \langle (\nabla h)^2 \rangle_c = 0$. Subtracting this equation from (3) yields $\delta W = -\delta r + u\langle n^{1+\rho} \rangle$, where $\delta r \equiv r_c - r$, and $\delta W \equiv \langle (\nabla h)^2 \rangle - \langle (\nabla h)^2 \rangle_c$. From the standard scaling of the KPZ equation [6], one has [10] $\delta W \sim -C \xi^{2(\chi-1)}$, where C is a positive constant and χ the roughness exponent of a KPZ interface; i.e., $\langle [h(\vec{x}, t) - h(\vec{0}, t)]^2 \rangle \sim x^{2\chi}$ for KPZ in steady state. Since $C > 0$, $\delta r > 0$ in the active phase, and $\langle n^{1+\rho} \rangle \sim (\delta r)^{\beta_{1+\rho}} > 0$, the equation for ξ has a solution for small δr only if the exponent $\beta_{1+\rho}$ is greater than unity. For the quadratic nonlinearity, $\rho = 0$, so $\langle n^{1+\rho} \rangle$ is the order parameter $\langle n \rangle$, whereupon this constraint places the nontrivial bound $\beta > 1$ on the order parameter exponent $\beta \equiv \beta_1$. For any value of ρ, one then obtains the result $\xi \sim \delta r^{-\nu}$ with $\nu = 1/(2 - 2\chi)$. (When the KPZ interface is smooth, $\chi = 0$ in this formula.) Note that the KPZ scaling relation $z + \chi = 2$ [6] implies $z - 1/2\nu = 1$ for MNS.

We now describe the numerical simulation of (1) in 1D. We discretize the continuum equation as

$$\begin{aligned} n_i(t + \Delta t) = n_i(t) + \Delta t\{ & -r n_i(t) - u n_i(t)^{2+\rho} \\ & + (1/\Delta x^2)[n_{i+1}(t) + n_{i-1}(t) - 2n_i(t)]\} \\ & + \sqrt{D\Delta t}\, n_i(t)\eta_i(t), \quad (i = 1, 2, \ldots, N), \end{aligned} \tag{4}$$

where Δx is the lattice spacing, Δt is the time step, and $\eta_i(t)$ is a Gaussian random number with unit standard deviation. [We have written Eq. (4) in the Ito interpretation [1] of Eq. (1). By replacing r by $r - D/2$ in all our results, one converts to the Stratonovich representation [1].] We set $\Delta x = 1$, $\Delta t = 0.02$, and use periodic boundary conditions with the system size $L = N\Delta x$. We also fix the noise amplitude $\sqrt{D} = 4$ and vary the linear coefficient r as the only control parameter. In the following, we present our numerical results for $\rho = 1$ [11].

In numerical simulation, due to the finite time step Δt, the property of (1) that $n(x, t) > 0$ if $n(x, 0) > 0$ for all x is lost. However, it is easy to fix this problem by setting $n_i(t)$ to zero if its value becomes negative under Eq. (4). The effect of this modification is easily estimated. For a single time step we can neglect the second term on the right-hand side (rhs) of (4) because it is of higher order in Δt than the noise term. Then setting negative values of $n_i(t)$ to 0 is equivalent to truncating the probability distribution of $\eta_i(t)$ so that its minimum is set by $\eta_{\min} = -1/\sqrt{D\Delta t}$, and replacing all the $\eta < \eta_{\min}$ by $\eta_{\min}$. This means, however, that the mean of $\eta_i(t)$ is no longer zero, and thus a deterministic term proportional to $n_i(t)$ is generated. The resulting effective linear coefficient r_{eff} can be roughly estimated as $r_{\text{eff}} = r - \sqrt{D/2\pi\Delta t} \int_{-\infty}^{\eta_{\min}} (\eta_{\min} - \eta) \exp(-\eta^2/2) d\eta$. The effective strength of the noise is also changed because of the truncation. These changes in parameters should not, however, alter the universality class of the transition [12].

The first step in studying the critical behavior of MNS numerically is to locate the critical point. Starting with uniform initial conditions and letting the equation evolve long enough to reach steady state, we compute $M = \langle n_i(t) \rangle$ for different values of r, where $\langle\ \rangle$ denotes both spatial and temporal averages as well as averages over different independent runs. We average over between 2×10^5 and 6×10^6 time steps and up to 100 independent runs, depending on the system size. To handle finite-size effects, we studied different system sizes: $N = 100, 200, 400, 1000$. In Fig. 1(a) we show the dependence on the inverse system size $1/N$ of the critical point $r_c(N)$ defined by M first becoming numerically indistinguishable from zero in every run. Extrapolating the fitted line [13] to $N = \infty$ determines a critical value $r_c \approx -2.18$. Figure 1(b) shows M vs $\delta r \equiv r_c - r$ on a log-log plot for $N = 400$. The best fit to $M \sim (\delta r)^{\beta}$ yields [14] $\beta = 1.70 \pm 0.05$.

Also depicted in Fig. 1(b) are the higher order moments of the n field: $M_m = \langle n^m \rangle$, with $m = 2, 3$. Evidently

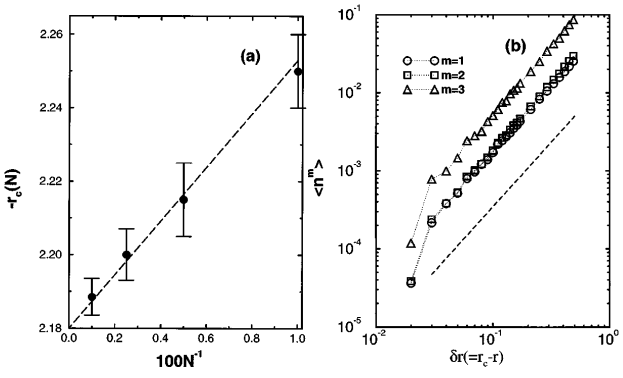

FIG. 1. (a) $r_c(N)$ vs inverse system size $100/N$; (b) $\langle n^m \rangle$ with $m = 1, 2, 3$ vs control parameter $\delta r \equiv r_c - r$; dashed line with slope 1.7 gives the best fit to the data.

M_2 and M_3 also have a power law dependence on δr: $M_m \sim (\delta r)^{\beta_m}$ with β_2 and β_3 being equal to β within our numerical accuracy. It is not surprising that there is anomalous scaling, i.e., $\beta_2 \neq 2\beta$, $\beta_3 \neq 3\beta$, and so on, since the strong coupling fixed point is non-Gaussian, but it remains to be seen whether β_m is indeed independent of m. It is interesting to note that for the zero-dimensional (single-variable) case, where an exact solution is available [3,15], all the moments do scale with exactly the same exponent: $\langle n^m \rangle = 2^{m/2}\Gamma[\frac{m+r/D}{2}]/\Gamma(-r/2D)$, i.e., $\beta_m = 1$ for all m.

To determine the other critical exponents, we need to calculate the two-point function $C(x,t) = \langle n(x + x_0, t + t_0) n(x_0, t_0) \rangle$ in the active phase. We have computed both $C(x,0)$ and $C(0,t)$, for $-r = 2.8, 2.6, 2.5, 2.4, 2.34, 2.3, 2.2, 2.215$. We used $N = 512$ and averaged over $T = 10^6 \Delta t$ time steps in steady state, and over up to 100 independent runs.

We summarize the results in Fig. 2. In Figs. 2(a) and 2(b), $C(x,0)$ and $C(0,t)$ are plotted for different values of r. It is clear from these figures that the scaling regime becomes bigger as we approach the critical point and that the amplitudes of the correlation functions become smaller, consistent with $C(x,t)$ vanishing, as it must, in the absorbing phase. In Fig. 2(c) we plot the absolute value of the *connected* space and time correlation function $C_c(x,t) = C(x,t) - M^2$ at the largest value of r, viz., -2.215. The power law decay of the correlation functions in the scaling regime can be characterized by $C_c(0,t) = A_t(r) t^{-\alpha_t}$ and $C_c(x,0) = A_x(r) x^{-\alpha_x}$ with the exponents $\alpha_t = 1.08 \pm 0.04$ and $\alpha_x = 1.65 \pm 0.07$. In conventional notation this corresponds to exponents $\eta \equiv 2 - d + \alpha_x = 2.65 \pm 0.07$ and $z \equiv \alpha_x/\alpha_t = 1.53 \pm 0.07$. In Fig. 2(d) we plot the dependence of the amplitudes of the spatial and temporal correlation functions, $A_x(r)$ and $A_t(r)$, on δr in the scaling regime; these can be fitted by $A_{x,t}(r) \sim \delta r^{\Delta}$ with the exponent $\Delta = 1.7 \pm 0.07$.

From these measured values of the four exponents β, η, z, and Δ, we obtain the correlation length exponent ν from the scaling relation [5] $\nu = (2\beta - \Delta)/(d - 2 + \eta) = 1.03 \pm 0.05$. These values of ν and z are in excellent agreement with the values 1 and $\frac{3}{2}$, respectively, following from our argument that the critical behavior of MNS is controlled by the KPZ equation; β_2 is also >1, consistent with the bound derived from KPZ. To further check the accuracy of various scaling exponents, we have also measured the decay of the average density right at the critical point, starting from a homogeneous initial condition $M \sim t^{-\theta}$. Using scaling arguments it is easy to express θ in terms of other exponents: $\theta = \beta/(\nu z)$. From the numerical values of β, ν, and z, we predict θ to be 1.079. In Fig. 3 we plot M versus time at $r = r_c$, and the exponent θ thus measured is $\theta = 1.1 \pm 0.05$, in excellent agreement with the scaling prediction. In the same figure we have included the behavior of the higher order moments $M_{2,3,4}(t)$ at the critical point. These plots strongly suggest that the exponents $\theta_m = \beta_m/(\nu z)$ for these higher moments are all equal to θ, in agreement with the static measurement.

Another important characterization of any phase transition is the response function. Though in equilibrium systems the fluctuation-dissipation theorem relates the response function to the correlation function, in nonequilibrium systems the response and correlation functions can differ significantly. Indeed, one of the striking features of MNS is that the susceptibility is predicted [5] to diverge over a finite range of control parameters, while the correlation function has, more conventionally, a unique critical point.

To test this prediction numerically in 1D, we add a constant source term ϕ on the rhs of Eq. (4) and calculate

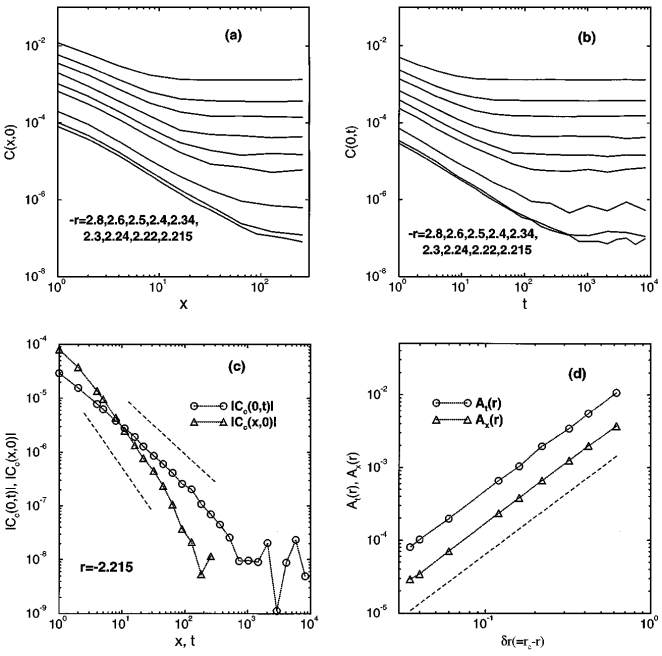

FIG. 2. (a) and (b) show the spatial and temporal correlation functions with different values of $r < r_c$; (c) *connected* correlation functions at $r = -2.215$; dashed lines are fits with slopes 1.08 and 1.65; (d) amplitudes of the correlation functions vs δr; dashed line has slope 1.7.

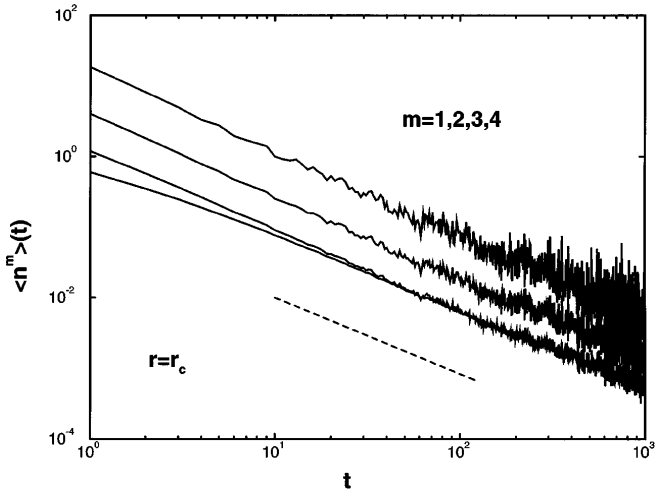

FIG. 3. Power-law decay in time of various moments of the order parameter at the critical point $r = r_c$.

the average density $M(\phi, r)$ for small values of ϕ and for each value of r. In Fig. 4(a) we show the dependence of $M(\phi, r)$ on ϕ for different values of r around the critical point r_c. We can fit the curves for small ϕ by $M(\phi, r) = M(0, r) + B(r)\phi^{\gamma(r)}$ for small ϕ, where $M(0, r)$ is the order parameter in the absence of any external source, and $B(r)$ is a function of r. The static susceptibility is defined as $\chi \equiv \lim_{\phi\to 0} \partial M/\partial\phi = \lim_{\phi\to 0} B(r)\gamma(r)\phi^{\gamma(r)-1}$, so the susceptibility diverges when $\gamma(r) < 1$. According to Fig. 4(b), where we show the dependence on r of the susceptibility exponent, $\gamma(r) - 1$, χ diverges not just in the absorbing phase, but in the whole region $3.2 > r > -5.9$ surrounding the critical point $r_c = -2.18$. Both the continuous variation of this exponent with r, and the divergence of χ in both the active and absorbing phases near r_c also occur in the exactly solvable 0D problem [5].

Another quantitative prediction of our previous paper [5] is that the region of diverging χ should terminate in the absorbing phase when the bare mass vanishes, i.e., at $r = 0$ in the Ito representation. However, because of the shift in r due to our numerical algorithm, the lower boundary of this region is shifted to $r > 0$. A clearer way to test the prediction is therefore to make the Hopf-Cole transformation $n = e^h$ first, and then simulate the equation for h, noting that the external field will have the form ϕe^{-h}. Because the Hopf-Cole relation automatically guarantees that the n field is positive definite, there is no need to alter the numerical algorithm, so $\gamma(r = 0)$ should equal 1. We have verified this in a simulation.

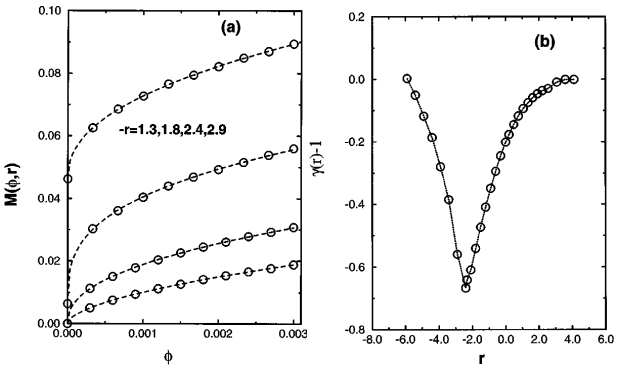

FIG. 4. (a) Order parameter vs external field at values of r near r_c; dashed lines are fits to $M(\phi, r) - M(0, r) \sim \phi^{\gamma(r)}$, as explained in text; (b) susceptibility exponent vs r; susceptibility diverges for a range of r values.

We thank A. Pikovsky, P. Grassberger, T. Hwa, S. Chen, and N.-Z. Cao for helpful discussions, and T. Hwa for bringing Refs. [9] to our attention.

[1] N. G. van Kampen, *Stochastic Processes in Chemistry and Physics* (North-Holland, Amsterdam, 1981).
[2] For example, *Percolation Structures and Processes*, edited by G. Deutsher, R. Zallen, and J. Adler, Annals of the Israel Physical Society Vol. 5 (Hiler, Bristol, 1980); J. Marro and R. Dickman, *Nonequilibrium Phase Transitions and Critical Phenomena* (Cambridge University Press, Cambridge, England, 1996), and references therein.
[3] A. Schenzle and H. Brand, Phys. Rev. A **20**, 1628 (1979).
[4] For example, W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984); C. Van den Broeck *et al.*, Phys. Rev. E **49**, 2639 (1994); C. Van den Broeck *et al.*, Phys. Rev. Lett. **73**, 3395 (1994).
[5] G. Grinstein, M. A. Muñoz, and Y. Tu, Phys. Rev. Lett. **76**, 4376 (1996).
[6] M. Kardar, G. Parisi, and Y. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[7] For example, Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).
[8] (a) A. Pikovsky and J. Kurths, Phys. Rev. E **49**, 898 (1994); (b) P. Grassberger (to be published); P. Grassberger and A. Pikovsky (unpublished).
[9] C. A. Doty and J. M. Kosterlitz, Phys. Rev. Lett. **69**, 1979 (1992); E. Frey and U. C. Tauber, Phys. Rev. E **50**, 1024 (1994).
[10] J. Krug and P. Meakin, J. Phys. A **23**, L987 (1990).
[11] Our simulations for $\rho = 0$ produced strong coupling exponents indistinguishable from those for $\rho = 1$.
[12] In problems of the directed percolation type [2], the same truncation would drastically change the problem, so more subtle discretization methods are required; see R. Dickman, Phys. Rev. E **50**, 4404 (1994).
[13] Standard finite-size scaling predicts $r_c \sim N^{-1/\nu}$, so r_c scaling like $1/N$ is consistent with $\nu = 1$.
[14] Numerical calculations for 1D coupled map models believed to be in the same universality class as Eq. (1) also give a β close to 1.70; see Ref. [8(b)].
[15] R. Graham and A. Schenzle, Phys. Rev. A **25**, 1731 (1982).