Static Axially Symmetric Solutions of Einstein–Yang-Mills-Dilaton Theory

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We construct static axially symmetric solutions of SU(2) Einstein–Yang-Mills-dilaton theory. Like their spherically symmetric counterparts, these solutions are nonsingular and asymptotically flat. The solutions are characterized by the winding number n and the node number k of the gauge field functions. For fixed n with increasing k the solutions tend to "extremal" Einstein-Maxwell-dilaton black holes with n units of magnetic charge. [S0031-9007(97)02792-0]

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SU(2) Einstein-Yang-Mills (EYM) theory possesses regular static spherically symmetric solutions [1]. These solutions are aymptotically flat and have nontrivial magnetic gauge field configurations, but no global charge. They have a high-density interior region, followed by a near-field region with approximately Reissner-Nordstrøm metric and a far-field region with approximately Schwarzschild metric [1].

To every regular solution in SU(2) EYM theory, there exists a corresponding family of black hole solutions with regular event horizon $x_{\rm H} > 0$ [2]. Outside their event horizon these black hole solutions possess non-Abelian hair. Like the regular solutions, the black hole solutions are unstable [3].

Like EYM theory, Einstein–Yang-Mills-dilaton (EYMD) theory possesses static, spherically symmetric, non-Abelian regular and black hole solutions [4]. Here the dilaton coupling constant γ represents a parameter. In the limit $\gamma \rightarrow 0$ the dilaton decouples and EYM theory is obtained, for $\gamma = 1$ contact with the low energy effective action of string theory is made, and in the limit $\gamma \rightarrow \infty$ gravity decouples and Yang-Mills-dilaton (YMD) theory is obtained.

Recently we have shown that YMD theory possesses also static axially symmetric regular solutions [5]. These solutions are labeled by the winding number $n > 1$ and the node number k of the gauge fields. For $n = 1$ one obtains the static spherically symmetric solutions of YMD theory [6]. The axially symmetric solutions have a toruslike shape. Choosing the z axis as the symmetry axis, the energy density has a strong peak along the ρ axis and decreases monotonically along the z axis.

The existence of the regular static axially symmetric YMD solutions strongly suggests the existence of the corresponding EYMD solutions for finite values of the dilaton coupling constant. In this Letter we present strong (numerical) evidence that such regular gravitating axially symmetric solutions indeed exist in EYMD theory and also in EYM theory.

Axially symmetric ansatz.—We consider the SU(2) Einstein–Yang-Mills-dilaton action

$$S = \int \left(\frac{R}{16\pi G} + L_M \right) \sqrt{-g}\, d^4x \,, \qquad (1)$$

with

$$L_M = -\tfrac{1}{2}\, \partial_\mu \Phi \partial^\mu \Phi - e^{2\kappa\Phi}\, \tfrac{1}{2}\, {\rm Tr}\, (F_{\mu\nu} F^{\mu\nu}), \qquad (2)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]$, and e and κ are the Yang-Mills and dilaton coupling constants, respectively.

To obtain static axially symmetric solutions we employ isotropic coordinates and adopt the metric

$$ds^2 = -f dt^2 + \frac{m}{f}\,(d\rho^2 + dz^2) + \frac{l}{f}\,\rho^2 d\phi^2, \qquad (3)$$

with f, m, and l being only functions of ρ and z. The corresponding ansatz for the purely magnetic gauge field ($A_0 = 0$) is [5,7]

$$A_\rho = \tfrac{1}{2}\, \tau^n_\phi w^3_1 \,, \qquad (4)$$

$$A_z = \tfrac{1}{2}\, \tau^n_\phi w^3_2 \,, \qquad (5)$$

$$A_\phi = \tfrac{1}{2}\, \tau^n_\rho \rho w^1_3 + \tfrac{1}{2}\, \tau_z \rho w^2_3 \,, \qquad (6)$$

with the Pauli matrices $\vec\tau = (\tau_x, \tau_y, \tau_z)$ and $\tau^n_\rho = \vec\tau \cdot (\cos n\phi, \sin n\phi, 0)$, $\tau^n_\phi = \vec\tau \cdot (-\sin n\phi, \cos n\phi, 0)$. The four functions w^i_j and the dilaton function Φ depend only on ρ and z.

Denoting the stress-energy tensor of the matter fields by T^ν_μ, with this ansatz the energy density $\epsilon = -T^0_0 = -L_M$ becomes

$$\begin{aligned} -T^0_0 = {}& \frac{f}{2m}\,[(\partial_\rho\Phi)^2 + (\partial_z\Phi)^2] + e^{2\kappa\Phi}\,\frac{f^2}{2m^2}\,(\partial_\rho w^3_2 - \partial_z w^3_1)^2 \\ & + e^{2\kappa\Phi}\,\frac{f^2}{2ml}\Bigg[\left(\partial_\rho w^1_3 + \frac{(n w^3_1 + w^1_3)}{\rho} - e w^3_1 w^2_3\right)^2 + \left(\partial_\rho w^2_3 + \frac{w^2_3}{\rho} + e w^3_1 w^1_3\right)^2 \\ & + \left(\partial_z w^1_3 + \frac{n w^3_2}{\rho} - e w^3_2 w^2_3\right)^2 + (\partial_z w^2_3 + e w^3_2 w^1_3)^2\Bigg]. \end{aligned} \qquad (7)$$

The system possesses a residual Abelian gauge invariance [5,8,9]. With respect to the transformation

$$U = e^{i\Gamma(\rho,z)\tau_\phi^n}, \tag{8}$$

the functions $(\rho w_3^1, \rho w_3^2 - n/e)$ transform like a scalar doublet, and the functions (w_1^3, w_2^3) transform like a two-dimensional gauge field. We fix the gauge by choosing the gauge condition [5,8,9]

$$\partial_\rho w_1^3 + \partial_z w_2^3 = 0. \tag{9}$$

To make contact with the spherically symmetric case $n = 1$, we introduce the coordinates r and θ ($\rho = r\sin\theta$, $z = r\cos\theta$) and the gauge field functions $F_i(r,\theta)$ [5,9]

$$w_1^3 = \frac{1}{er}(1 - F_1)\cos\theta, \quad w_2^3 = -\frac{1}{er}(1 - F_2)\sin\theta,$$
$$w_3^1 = -\frac{n}{er}(1 - F_3)\cos\theta, \quad w_3^2 = \frac{n}{er}(1 - F_4)\sin\theta. \tag{10}$$

The spherically symmetric ansatz of Ref. [4] is recovered for $F_1(r,\theta) = F_2(r,\theta) = F_3(r,\theta) = F_4(r,\theta) = w(r)$, $\Phi(r,\theta) = \phi(r)$, and $n = 1$.

The above ansatz and gauge choice yield a set of coupled partial differential equations for the metric and the matter field functions. To obtain regular asymptotically flat solutions with finite energy density we impose at the origin ($r = 0$) the boundary conditions

$$\partial_r f = \partial_r m = \partial_r l = \partial_r \Phi = 0,$$
$$F_1 = F_2 = F_3 = F_4 = 1, \tag{11}$$

and at infinity ($r = \infty$)

$$f = m = l = 1, \quad \Phi = 0,$$
$$F_1 = F_2 = F_3 = F_4 = \pm 1; \tag{12}$$

further we impose for all functions that their derivatives with respect to θ vanish on the ρ and the z axes [5]. The boundary conditions for the gauge field functions at infinity imply that the solutions are magnetically neutral. Note that a finite value of the dilaton field at infinity can always be transformed to zero via $\Phi \to \Phi - \Phi(\infty)$, $r \to re^{-\kappa\Phi(\infty)}$.

The mass M of the regular axially symmetric solutions can be obtained directly from the total energy-momentum "tensor" $\tau^{\mu\nu}$ of matter and gravitation, $M = \int \tau^{00} d^3r$ [10], or from $M = -\int (2T_0^0 - T_\mu^\mu)\sqrt{-g}\, dr\, d\theta\, d\phi$. Both expressions give the same values for the mass of the solutions.

We now remove the dependence on the coupling constants G and e from the differential equations by changing to the dimensionless coordinate $x = (e/\sqrt{4\pi G})r$, the dimensionless dilaton function $\varphi = \sqrt{4\pi G}\,\Phi$, and the dimensionless coupling constant $\gamma = \kappa/\sqrt{4\pi G}$ ($\gamma = 1$ corresponds to string theory). The dimensionless mass is then given by $\mu = (e/\sqrt{4\pi G})GM$.

In the spherically symmetric case the following relations between the metric and the dilaton field hold [11]

$$\varphi(x) = \tfrac{1}{2}\gamma \ln(-g_{tt}), \tag{13}$$

$$\mu = \frac{1}{2}x^2 \partial_x f|_\infty = \frac{1}{\gamma}x^2\partial_x\varphi|_\infty = \frac{D}{\gamma}, \tag{14}$$

where D is the dilaton charge. These relations also hold for the regular axially symmetric solutions considered here. Their derivation is based on the equation of motion of the dilaton field and will be given elsewhere [12].

Solutions.—Subject to the above boundary conditions, we solve the equations numerically. To map spatial infinity to the finite value $\bar{x} = 1$, we employ the radial coordinate $\bar{x} = x/(1 + x)$. The numerical calculations are based on the Newton-Raphson method [13]. The equations are discretized on a nonequidistant grid in $\bar{x}$ and θ. Typical grids used have sizes 150×30, covering the integration regions $0 \le \bar{x} \le 1$ and $0 \le \theta \le \pi/2$. The numerical error for the functions is estimated to be on the order of 10^{-3} and 10^{-2} for $k < 4$ and $k = 4$, respectively.

In Tables I and II we show the dimensionless mass of a subset of the regular axially symmetric solutions obtained so far. The energy density $\epsilon = -T_0^0$ of the solutions has a strong peak along the ρ axis, and it decreases monotonically along the z axis. Thus equal density contours reveal a toruslike shape of the solutions. As a typical example we show the energy density ϵ of the solution with $n = 3$, $k = 3$, and dilaton coupling constant $\gamma = 1$ in Fig. 1.

With n and γ fixed and increasing k, the location of the peak of the energy density moves inward and the peak increases in height, whereas with fixed k and γ and increasing n the peak of the energy density moves outward and decreases in height. This is demonstrated in Table II.

The gauge field functions F_i and the dilaton function φ of the regular axially symmetric EYMD and EYM solutions look similar to those of the corresponding YMD solutions [5]. Like the dilaton function φ, the metric

TABLE I. The dimensionless mass μ of the EYMD solutions with winding number $n = 3$ and up to 4 nodes for several values of the dilaton coupling constant γ. For comparison, the last row gives the mass of the limiting solutions, the first column gives the mass of the EYM solutions, and the last column the scaled mass of the corresponding YMD solutions [5].

	EYM		EYMD		YMD
k/γ	0	0.5	1.0	2.0	∞
1	1.870	1.659	1.297	0.811	1.800
2	2.524	2.250	1.770	1.114	2.482
3	2.805	2.505	1.976	1.247	2.785
4	2.922	2.611	2.063	1.304	2.913
…	3	2.683	2.121	1.342	3

TABLE II. The dimensionless mass μ, the maximum of the energy density $\epsilon_{\max}$ and the location $\rho_{\max}$ of the maximum of the energy density of the EYMD solutions of the sequences $n = 1-4$ with node numbers $k = 1-4$ and $\gamma = 1$. For comparison, the mass of the limiting solutions is also shown. Note, that μ/n decreases with n for fixed finite k.

	μ			
k/n	1	2	3	4
1	0.577	0.961	1.297	1.607
2	0.685	1.262	1.770	2.239
3	0.703	1.365	1.976	2.549
4	0.707	1.399	2.063	2.698
−	0.707	1.414	2.121	2.828
	$\epsilon_{\max}(\rho_{\max})$			
k/n	1	2	3	4
1	1.075 (0.)	0.177 (0.90)	0.098 (1.59)	0.072 (2.37)
2	11.63 (0.)	0.910 (0.30)	0.380 (0.66)	0.235 (1.10)
3	79.70 (0.)	3.443 (0.09)	1.124 (0.28)	0.601 (0.51)
4	498.2 (0.)	12.01 (0.03)	3.064 (0.11)	1.435 (0.24)

functions do not exhibit a strong angular dependence. These functions will be exhibited elsewhere [12].

For fixed n and γ, with increasing k the sequence of axially symmetric solutions tends to a limiting solution. The gauge field functions $(F_i)_k$ tend to the limiting function $F_\infty = 0$. (Because of the boundary conditions at the origin and at infinity, they approach the limiting function nonuniformly.) The dilaton functions φ_k tend to the limiting function φ_∞, which represents the dilaton function of the "extremal" EMD solution [14] with n units of magnetic charge and the same value of γ. This is demonstrated in Fig. 2 for $n = 3$ and $\gamma = 1$. We observe that φ_k deviates from φ_∞ only in an inner region, which decreases exponentially with k.

For finite values of γ and fixed n, the sequences of solutions thus approach as limiting solutions the extremal EMD solutions [14] with magnetic charge n and the same γ. This generalizes the corresponding observation for the spherically symmetric solutions with $n = 1$ [4,11]. For $\gamma = 0$ the limiting solutions represent the exterior of extremal Reissner-Nordstrøm black holes with charge n. For the YMD solutions the limiting solutions are magnetic monopoles with n units of charge [5].

The limiting values for the mass, $\mu = n/\sqrt{1+\gamma^2}$ [14], represent upper bounds for the sequences, as observed from Tables I and II. The larger n, the slower is the convergence to the limiting solution. Further details on the convergence properties of the sequences of solutions will be given elsewhere [12].

In addition to the known static spherically symmetric solutions, both EYMD and EYM theory possess sequences of regular static axially symmetric solutions. These sequences are characterized by the winding number $n > 1$, and the solutions within each sequence by the node number k. (For $n = 1$ the spherically symmetric solutions are recovered.) For fixed n and γ, with increasing k the solutions tend to the "extremal" Einstein-Maxwell-Dilaton solution [14] with n units of magnetic charge and the same γ.

The multisphalerons have a toruslike shape. Apart from that, many properties of the axially symmetric solutions are similar to those of their spherically symmetric counterparts. In particular, there is all reason to believe that these regular static axially symmetric EYMD and EYM solutions are unstable. Since we can also associate the Chern-Simons number $N_{CS} = n/2$ [9] (for odd k [15]) with these solutions, we interpret them as *gravitating multisphalerons.*

Having constructed the axially symmetric solutions in EYMD and EYM theory, it appears straightforward to construct analogous solutions in theories with a Higgs field. We therefore expect to find gravitating axially symmetric multimonopoles (for the case of a Higgs triplet) or gravitating axially symmetric multisphalerons

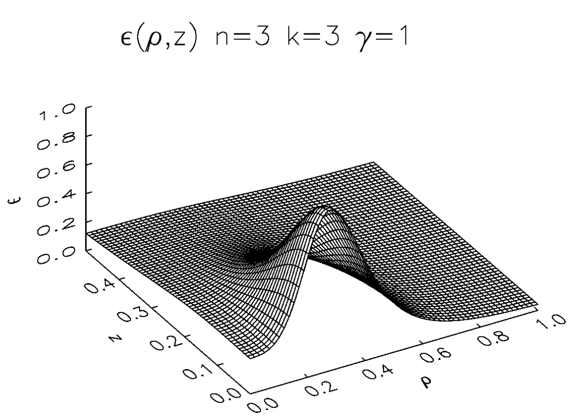

FIG. 1. The energy density $\epsilon = -T_0^0$ is shown as a function of the dimensionless coordinates ρ and z for the EYMD solution with winding number $n = 3$, node number $k = 3$, and $\gamma = 1$.

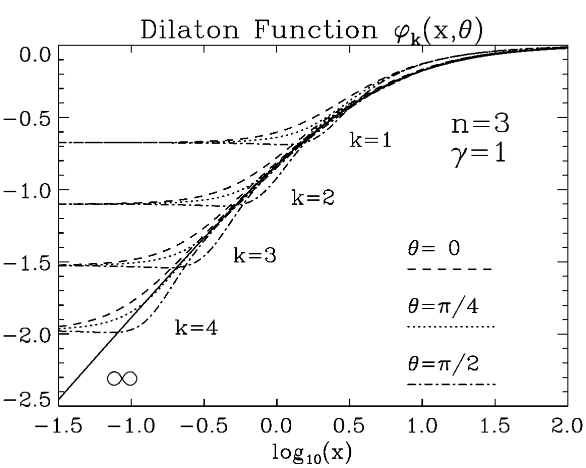

FIG. 2. The dilaton functions $\varphi_k(x,\theta)$ for the EYMD solutions with winding number $n = 3$, node numbers $k = 1-4$, and $\gamma = 1$ are shown as a function of the dimensionless coordinate x. The dashed, the dotted, and the dash-dotted lines represent the angles $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$, respectively. Also shown is the limiting function $\varphi_\infty(x)$ (solid line).

(for the case of a Higgs doublet). Similarly there should be gravitating axially symmetric multi-Skyrmions. Work along these lines is in progress.

We consider the above set of solutions to be the simplest type of gravitating nonspherical regular solutions of EYMD and EYM theory. We conjecture that there are gravitating regular solutions with much more complex shapes and only discrete symmetries left. This conjecture is based on the observation that for some types of solitons in flat space the symmetry structure of the (energetically lowest) solutions becomes increasingly complex with increasing winding number or charge n. For instance, for Skyrmions, the lowest $n = 1$ solution is spherically symmetric, the lowest $n = 2$ solution has axial symmetry, and the lowest $n \geq 3$ solutions respect only discrete crystal-like symmetries [16].

But EYMD and EYM theory also possess black hole solutions. The non-Abelian spherically symmetric black hole solutions may be regarded as black holes inside sphalerons [15]. We conjecture that also the gravitating axially symmetric solutions can accommodate black holes in their interior. And this conjecture naturally extends to gravitating solutions with more complex shapes and less symmetry. The existence of such black hole solutions without rotational symmetry inside multimonopoles has also been conjectured from a stability argument [17].

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