Quantum Fluctuations and Dynamical Chaos

Sergei G. Matinyan[1,2] and Berndt Müller[1]
[1]*Department of Physics, Duke University, Durham, North Carolina 27708-0305*
[2]*Yerevan Physics Institute, Yerevan, Armenia*
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We discuss the intimate connection between the chaotic dynamics of a classical field theory and the instability of the one-loop effective action of the associated quantum field theory. Using massless scalar electrodynamics as an example, we show how the radiatively induced spontaneous symmetry breaking stabilizes the vacuum state against chaos. [S0031-9007(97)02869-X]

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It is generally believed that quantum fluctuations lead to the suppression of the most characteristic manifestations of dynamical chaos [1]. For the mechanical systems it is obvious: the discreteness of the phase space imposed by quantum mechanics suppresses or even eliminates the long-time behavior of the classically chaotic systems that is characterized by the positivity of the Lyapunov exponents. Indeed, the nonstationary evolution of a classically chaotic quantum mechanical system (scalar electrodynamics) is characterized by vanishing Lyapunov exponents [2].

For field theories with their infinite number of degrees of freedom the situation is not so straightforward. It is well established that not only the spatially uniform limits of Yang-Mills theory and scalar electrodynamics but also various field theories, among which are the spherically symmetric Yang-Mills equations [3], the Yang-Mills-Higgs equations in the interior of a 't Hooft-Polyakov monopole [4], and the equations of general relativity [5,6], exhibit dynamical chaos in the classical limit (see [7] for details and references). Here, as in the case of the mechanical systems, the basic question of the competition and interference between the highly unstable classical fluctuations responsible for chaos and the quantum fluctuations of the interacting fields arises. Do the quantum fluctuations suppress the chaoticity of the classical field theory? Although practically all methods of the quantization of fields about chaotic classical solutions encounter this problem of instability there does not exist a proven way to avoid or circumvent this delicate problem [8,9].

In this Letter, we do not propose a general recipe for the quantization of field theories that are chaotic in the classical limit, but confine ourselves to the loop expansion taking as a basis the chaotic classical theory. Our treatment is based on the notion of the effective potential. We consider here, as an example, mostly massless scalar electrodynamics since it is free from the well-known difficulties arising when, in spontaneous symmetry breaking, the new minimum lies far outside the validity of the one-loop approximation [10]. As we will show, the quantum corrections to the classical potential increase the threshold for chaos by modifying the ground state of the system. The chaoticity of the classical field, in turn, is reflected in the presence of an imaginary part of the effective potential in the loop expansion.

Now, we briefly describe the chaotic properties of classical scalar electrodynamics defined by the Lagrangian density for the system with bare scalar mass m_0:

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m_0^2\phi^*\phi\,, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu + ieA_\mu$. We consider the case of spatially homogeneous classical fields $A_\mu(t)$ and $\phi(t)$ for which the study of chaos is extremely simplified. In the gauge $A_0 = 0$, Gauss' law implies that the phase of the scalar field $\phi = \frac{1}{\sqrt{2}}\rho e^{i\alpha}$ is time independent. Taking only a single component of A_i nonvanishing, we arrive at the following simple system of equations [11,12]:

$$\ddot{\rho} + (m_0^2 + e^2A^2)\rho = 0, \qquad \ddot{A}_i + e^2\rho^2 A_i = 0\,, \quad (2)$$

where dots indicate time derivatives. This system is classically equivalent to the well-known two-dimensional dynamical system with the quartic potential x^2y^2 exhibiting a strongly chaotic behavior. This system appears in various contexts in science including chemistry, astronomy, astrophysics, cosmology, and most interesting for us, in the free Yang-Mills equations [7,13].

The stability of the motion of the system (2) with the quartic potential

$$U_{\rho A} = \frac{1}{2}m_0^2\rho^2 + \frac{1}{2}e^2\rho^2A^2, \quad (3)$$

and separable, quadratic kinetic energy is determined by the time-dependent eigenvalues of the 4×4 stability matrix

$$\begin{pmatrix} 0 & -U''_{\rho A} \\ 1 & 0 \end{pmatrix}, \quad (4)$$

where 0 and 1 are the 2×2 null and unit matrices, respectively, and the 2×2 matrix $U''_{\rho A}$ has the form

$$U''_{\rho A} = \begin{pmatrix} m_0^2 + e^2A^2 & 2e^2\rho A \\ 2e^2\rho A & e^2\rho^2 \end{pmatrix}. \quad (5)$$

Hence, if the matrix (5) has a negative eigenvalue for (almost) any time, the system is unstable [14]. In other words, the criterion for dynamical instability of our system is that

$$\det\{U''_{\rho A}\} < 0 \tag{6}$$

for (almost) any time. The condition (6) immediately gives us the following condition for the onset of chaos:

$$e^2A^2 > e^2A^2_{\rm cr} = \frac{m_0^2}{3} \tag{7}$$

for any ρ, with the corresponding energy threshold for chaos

$$E_{\rm cr} = \frac{2}{3}m_0^2\rho^2 \tag{8}$$

at the classical minimum of the potential $E_{\rm cr} = 0$. Thus classical *massless* scalar electrodynamics is strongly chaotic in the long wavelength limit for any magnitude of the spatially homogeneous fields and for all values of the total energy. The mass m_0 of the scalar field sets a threshold for chaos.

The self-interaction $\lambda\phi^4$ $(\lambda > 0)$ of the scalar field increases the threshold for chaos. Indeed, adding to the potential (3) the quartic term $\frac{\lambda}{4}\rho^4$ we obtain

$$e^2A^2_{\rm cr} = \lambda\rho^2 + \frac{m_0^2}{3}. \tag{9}$$

At the classical minimum $\rho = 0$, still $E_{\rm cr} = 0$.

If the gauge symmetry is spontaneously broken at the classical level, i.e., for the potential

$$U_{\rho A} = \frac{1}{2}e^2A^2\rho^2 + \frac{\lambda}{4}(\rho^2 - v^2)^2 \tag{10}$$

with the vacuum expectation value of the scalar field v and its mass $m_s^2 = 2\lambda v^2$, we obtain the following critical magnitude of the gauge field $A_{\rm cr}$ for the onset of chaos:

$$e^2A^2_{\rm cr} = \lambda\left(\rho^2 - \frac{v^2}{3}\right). \tag{11}$$

At the minimum of the potential (10) this gives

$$e^2A^2_{\rm cr} = \frac{m_s^2}{3}, \tag{12}$$

which coincides with the condition (7) for the onset of chaos for the case of *massive* classical scalar electrodynamics without self-interaction.

Substituting $A^2_{\rm cr}$ into (10) and minimizing $U_{\rho A}$ with respect to ρ we find [15]

$$E_{\rm cr}|_{\rho=v} = \frac{11}{108}\lambda v^4. \tag{13}$$

Turning now to the quantum corrections, we base our discussion on the effective potential approach. As is well known [10], radiative corrections induce a nontrivial minimum of the effective potential Γ, generating mass and thus increasing the threshold for chaos.

The one-loop effective potential for massless scalar electrodynamics can be written as

$$\Gamma^{(1)}(\rho, A) = U_{\rho A} + \frac{\hbar}{64\pi^2}{\rm tr}[(U''_{\rho A})^2\ln(U''_{\rho A}/\mu^2)], \tag{14}$$

where $U_{\rho A}$ is the classical potential (3) and the matrix $U''_{\rho A}$ is given by (5). Here μ^2 is the renormalization point.

From the definition of the effective potential, it is evident that the exact effective potential must be real. The approximate calculation of this quantity in the loop expansion leads to regions of complexity which are impossible to eliminate for the system under study. We see from (14) that for a classically chaotic system characterized by the condition (6) the one-loop effective potential becomes complex for almost all values of the fields A, ρ and not only for some finite range of the fields as it occurs, e.g., in the case of spontaneous symmetry breaking at the tree level for nonchaotic systems. Massless scalar electrodynamics and the free Yang-Mills theory in the limit of homogeneous fields are such chaotic systems.

It is possible to say that the complexity of the loop-expanded effective potential is a relic of the chaoticity of the classical theory. The imaginary part of the effective potential signals not only the instability of the field configuration, but it is a general consequence of the chaos of the classical system.

From these considerations one can understand why all efforts to eliminate the imaginary part [16] of the one-loop effective potential for the uniform chromomagnetic field of the pure Yang-Mills theory in Minkowski space [17] were unsuccessful. Stable radiative corrections in Minkowski space require a stable classical configuration. The presence of the imaginary part in the one-loop effective potential is intrinsically linked to the asymptotic freedom of the non-Abelian gauge fields [16,18]. It is worth noting that recently this unstable mode was detected directly in Monte Carlo simulations of the lattice gauge theory [19].

In order to show how the quantum fluctuations set the threshold for the onset of chaos and even suppress it, we temporarily ignore the imaginary part of the potential (14) and, following Ref. [20], consider only the effect of the quantum corrections along the axis $A = 0$. This "projection" retains the picture of spontaneous symmetry breaking by the quantum corrections, because the actual minima of the real part of $\Gamma^{(1)}$ occur on the axes $A = 0$ and $\rho = 0$, as a numerical evaluation of the one-loop effective action in the ρ-A plane shows.

At this stage, we turn to consider the one-loop corrected effective potential for massless scalar electrodynamics with quartic self-interaction of the scalar field. We begin with the case without spontaneous symmetry breaking at the classical level. The quantum corrections lead to a new minimum of the potential at $A = 0$ but $\rho = \bar{\rho} \neq 0$ instead of $A = \rho = 0$. Implementing the standard procedure of dimensional transmutation [10], we write the one-loop effective potential as

$$\Gamma^{(1)}(\rho, A; \bar{\rho}) = \frac{1}{2} e^2 A^2 \rho^2 + \frac{5\rho^4}{32\pi^2}\left(\lambda^2 + \frac{3e^4}{10}\right) \times \left[\ln \frac{\rho^2}{\bar{\rho}^2} - \frac{1}{2}\right], \quad (15)$$

where the ρ^4 term of the classical potential is absorbed in the subtraction point of the logarithm. The effective potential now has a minimum value

$$E_0^{(1)} \equiv \Gamma^{(1)}(\rho, A; \bar{\rho})|_{\substack{\rho=\bar{\rho} \\ A=0}} = -\frac{5\bar{\rho}^4}{64\pi^2}\left(\lambda^2 + \frac{3e^4}{10}\right), \quad (16)$$

which lies below the classical vacuum. The masses of the scalar boson and photon are

$$m_s^2 = \frac{\partial^2 \Gamma^{(1)}}{\partial \rho^2}\bigg|_{\substack{\rho=\bar{\rho} \\ A=0}} = \delta m_\lambda^{(1)^2} + \delta m_e^{(1)^2}, \quad (17)$$

$$m_A^2 = \frac{\partial^2 \Gamma^{(1)}}{\partial A^2}\bigg|_{\substack{\rho=\bar{\rho} \\ A=0}} = e^2 \bar{\rho}^2, \quad (18)$$

where

$$\delta m_\lambda^{(1)^2} = \frac{5\lambda^2}{4\pi^2}\bar{\rho}^2, \qquad \delta m_e^{(1)^2} = \frac{3e^4}{8\pi^2}\bar{\rho}^2 \quad (19)$$

are the one-loop mass corrections to the classically masslessscalar boson generated by the scalar self-coupling and scalar-photon coupling, respectively.

We now consider (15) for the case of spatially uniform fields $\rho(t)$, $A(t)$ and apply the criterion (6) obtaining the critical value of A beyond which the chaos sets in:

$$e^2 A_{\mathrm{cr}}^2 = \frac{5\rho^2}{8\pi^2}\left(\lambda^2 + \frac{3e^4}{10}\right)\left[\ln \frac{\rho^2}{\bar{\rho}^2} + \frac{2}{3}\right]. \quad (20)$$

Taking (20) in the vicinity of the new minimum $\rho = \bar{\rho}$, where our equations are reliable, we arrive at the relation

$$e^2 A_{\mathrm{cr}}^2 = \frac{m_s^2}{3}, \quad (21)$$

where m_s is given by (17). The comparison with (7) shows that quantum corrections generate a finite threshold for the onset of chaos in massless scalar electrodynamics, as opposed to the classical theory which is chaotic for an infinitesimal amplitude of the gauge field.

Let us briefly address the case of massless scalar electrodynamics with spontaneous symmetry breaking at tree level. Adding to the classical potential (10) the one-loop quantum corrections, one finds again that the quantum fluctuations increase the stability against chaos, over and above the stabilizing threshold (11) introduced at the tree level by the mass generated by spontaneous symmetry breaking.

The situation is repeated for the case of the two-loop effective potential, leading to the threshold value

$$e^2 A_{\mathrm{cr}}^2 = \frac{1}{3}(\delta m_e^{(1)^2} + \delta m_e^{(2)^2}) = \frac{m_s^2}{3}, \quad (22)$$

where $\delta m_e^{(2)^2}$ is the two-loop scalar boson mass correction [21] up to $O(e^6)$ in the limit $\lambda^2 \ll e^4$. Equations (21) and (22) are special cases of the general relation

$$3e^2 A_{\mathrm{cr}}^2 = \frac{\partial^2 \Gamma}{\partial \rho^2}\bigg|_{\substack{\rho=\rho_{\min} \\ A=0}}, \quad (23)$$

which expresses the specific structure of scalar electrodynamics and the x^2y^2 model revealed by (5).

Up to now we have avoided the complications arising from the imaginary part of the effective potential—appearing due to the chaoticity of the classical field theory—by "projecting" the quantum corrections onto the axis $A = 0$. The imaginary part of $\Gamma^{(1)}(\phi)$ has a physical interpretation [21–23] describing the decay rate per unit volume of the initial quantum state, or the damping rate of certain correlation functions. This interpretation of the imaginary part of $\Gamma^{(1)}$ allows one to understand the coincidence of the maximal Lyapunov exponents of the classically chaotic SU(2) and SU(3) gauge theories and the corresponding analytically calculated damping rates of excitations of the thermalized gauge fields [24].

In conclusion, we have shown that the onset of chaoticity of the classical fields in theories such as scalar electrodynamics is delayed by the radiative corrections. In the case of massless scalar electrodynamics, which is chaotic for all energies at the tree level, the quantum corrections introduce a finite threshold for the onset of chaos.

The classical chaoticity, in turn, leads to the instability of the effective potential in the quantum theory, presumably, at any finite order of the loop expansion. Since the true effective potential is known to be always a real and convex function of the field expectation values, the instabilities associated with deterministic chaos must be absent in the full quantum theory. Higher-order (non-Gaussian) quantum fluctuations can provide the mechanism for this phenomenon. Unfortunately, it is not known how to perform functional integrals in quantum field theory beyond the Gaussian approximation by analytical techniques.

This raises the question whether, in a given theory, it is possible to find a stable *classical* configuration in *Minkowski space* around which the theory can be quantized. Several mechanisms are known [7] which generate stable solutions and hence eliminate chaos at low energies in gauge theories: mass generation by the Higgs mechanism [13] or topological effects, mass generation by medium polarization at finite temperature [25], and stabilization of fluctuations by external charges [26,27]. Although none of these mechanisms directly applies to the QCD vacuum, the quark vacuum condensate may have a similar stabilizing effect. On a more general scope, the question of the possible stabilizing role of fermions in supersymmetric Yang-Mills theories arises. We hope to return to these questions in the future.

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[1] But not all of them: Important properties of stationary states such as the distribution of energy level or interference patterns (scars) reflect the intrinsic chaoticity of the corresponding classical systems [see, e.g., L. E. Reichl, *The Transition to Chaos: In Conservative Classical Systems: Quantum Manifestations* (Springer-Verlag, New York, 1992)].

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[8] One might be tempted to say that such theories should not be considered as a basis of a *local* quantum field theory, and confine oneself to integrable systems. Not sharing this opinion, we believe that this view reflects our inability to go beyond the saddle-point approximation to the functional integration by analytical means.

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