Effective Actions for Spin Ladders

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We derive a path-integral expression for the effective action in the continuum limit of an antiferromagnetic Heisenberg spin ladder with an arbitrary number of legs. The map is onto an O(3) nonlinear σ model (NLσM) with the addition of a topological term that is effective only for odd-leg ladders and half-odd integer spins. We derive the parameters of the effective NLσM and the behavior of the spin gap for the case of even-leg ladders. [S0031-9007(97)02716-6]

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Triggered by the discovery of high-T_c superconductivity [1] and by the fact that, in the strong-coupling regime, the Hubbard model maps onto an antiferromagnetic (AFM) Heisenberg model [2], quantum spin systems have become in recent years of great theoretical and experimental interest [3]. Independently from the possible connections with high-T_c materials, some years ago Haldane [4] put forward the conjecture that the value of the spin should discriminate dramatically between half-odd integer and integer one-dimensional spin chains: The former should be gapless, with power-law decay of spin-spin correlations, while the latter should be characterized by exponential decay of correlations and hence by a nonvanishing spin gap. It turns out [5] that, in the continuum limit, an AFM Heisenberg chain is described by an O(3) nonlinear σ model (NLσM) [6] with the addition of a topological term (a Pontrjagin index [6]), multiplied by a coefficient $\theta = 2\pi S$, where S is the value of the spin. As shown, e.g., by Shankar and Read [7], it is precisely this term that makes the half-odd integer spin chain (where $\theta = \pi \bmod 2\pi$) massless, while it is ineffective for integer spin chains (where $\theta = 0 \bmod 2\pi$), which are gapped.

Extensions of these results to $D = 2$ turned out to be rather disappointing, to the extent that it has been proved in a convincing way [8–12] that, at least for smooth configurations (see, however, [8] for the case of a singular "hedge-hog" field configuration), the topological term is absent irrespective of the value of the spin and also of the topology of the lattice.

Quite recently, however, it came as a surprise when theoretical (mostly numerical) and, subsequently, experimental results showed that spin *ladders*, obtained by antiferromagnetically coupling a finite number of chains, show drastically different behaviors that are basically determined by the value of the spin *and* by the number of legs in the ladder. Coupled spin chains had already been studied previously [13] mainly as models of single chains with spin higher than the minimum value $S = \frac{1}{2}$. A systematic study of the appearance of a spin gap in even-leg ladders and of the possible onset of superconductivity upon doping with holes was initiated later on by Dagotto *et al.* [14] who made for the first time quantitative predictions on both the spin gap and on the pair correlation function. Odd-leg ladders are instead gapless. This "even-odd" conjecture was formulated for the first time by Rice *et al.* [15], and is very much reminiscent of the already mentioned Haldane's "integer–half integer" conjecture [4] for single spin chains. According to this "even-odd" conjecture, undoped integer-spin ladders are gapped while half-odd integer spin ladders are gapped when the number of legs is even, gapless when it is odd. These results have by now strong support, both theoretically [16] and experimentally [17] (for a recent review, see [18]). The situation seems to be not so simple for conducting (i.e., hole-doped) ladders. A different behavior between even-leg and odd-leg ladders is predicted only for very strong intrachain repulsions [19]. On the contrary, in the weak coupling regime, analytical [19,20] and numerical [21] studies suggest that d-wave interchain pairing correlations become dominant over AF fluctuations in both the two-leg and the three-leg ladder, due to the simultaneous presence of gapless and gapped spin modes. So, the situation is richer than in the case of single conducting chains, that display a Luttinger liquid type [18,22] behavior.

Following mainly the ideas of Haldane [4] and Affleck [5] on the role of the topological term in spin chains, considerable theoretical effort has been recently devoted to the investigation of the existence and the role of such a term when a finite number of chains is coupled to form a ladder (the possible topological origin for the different behavior of even-leg and odd-leg ladders has been suggested for the first time in [23]). In particular, Sénéchal [24] has given a derivation of the NLσM continuum limit of a two-leg ladder using a coherent-state path-integral [25] expression for the partition function, while Sierra [26] has employed an (operator) Hamiltonian approach following closely Affleck's mapping of the Heisenberg chain onto the NLσM.

What we present here is a path-integral approach that is, however, different from Sénéchal's. It allows us to reproduce Sénéchal's results for the two-leg ladder, but also to generalize them to ladders with an arbitrary number of legs and to obtain a clear understanding of why the topological term is absent for even-leg ladders.

The Hamiltonian for a ladder system with n_l legs of length N is defined by

$$H = \sum_{a=1}^{n_l} \sum_{i=1}^{N} [J_a \boldsymbol{S}_a(i) \cdot \boldsymbol{S}_a(i+1) + J'_{a,a+1} \boldsymbol{S}_a(i) \cdot \boldsymbol{S}_{a+1}(i)]. \qquad (1)$$

The only condition we shall impose on the coupling constant J_a and $J'_{a,a+1}$ is that the classical minimum of the Hamiltonian (1) be antiferromagnetically ordered. The partition function for the Hamiltonian (1) in a path-integral representation which makes use of spin coherent states [25] is given by

$$\mathcal{Z}(\beta) = \int [D\hat{\boldsymbol{\Omega}}] \exp\left\{ is \sum_{i,a} \omega[\hat{\boldsymbol{\Omega}}_a(i,\tau)] - \int_0^\beta d\tau H(\tau) \right\}, \qquad (2)$$

where $\omega[\hat{\boldsymbol{\Omega}}_a(i,\tau)]$ is the Berry phase factor coming from the exponentiation of the overlap between coherent states at nearby time slices [27], while $H(\tau)$ is obtained by replacing in the Hamiltonian (1) the operator $\boldsymbol{S}_a(i)$ by the classical variable $s\hat{\boldsymbol{\Omega}}_a(i,\tau)$. Explicitly,

$$\omega[\hat{\boldsymbol{\Omega}}(\tau)] = \int_0^\beta d\tau \dot{\phi}(\tau)[1 - \cos\theta(\tau)], \qquad (3)$$

$$H(\tau) = \sum_{a,i} [J_a s^2 \hat{\boldsymbol{\Omega}}_a(i,\tau) \cdot \hat{\boldsymbol{\Omega}}_a(i+1,\tau) + J'_{a,a+1} s^2 \hat{\boldsymbol{\Omega}}_a(i,\tau) \cdot \hat{\boldsymbol{\Omega}}_{a+1}(i,\tau)]. \qquad (4)$$

The Berry phase measures the area enclosed by the path $\hat{\boldsymbol{\Omega}}(\tau) = (\sin\theta(\tau)\cos\phi(\tau), \sin\theta(\tau)\sin\phi(\tau), \cos\theta(\tau))$ on the unit sphere. We shall proceed now along the lines of Ref. [8]. We assume that short-range antiferromagnetic correlations survive at the quantum level, so that we consider the dominant contribution to the path integral as coming from paths described by

$$\hat{\boldsymbol{\Omega}}_a(i,\tau) = (-1)^{a+1} \hat{\boldsymbol{\phi}}(i,\tau) \left(1 - \frac{|\boldsymbol{l}_a(i,\tau)|^2}{s^2}\right)^{1/2} + \frac{\boldsymbol{l}_a(i,\tau)}{s}. \qquad (5)$$

The fluctuation field $\boldsymbol{l}_a(i)$ is supposed to be small: $|\boldsymbol{l}_a(i)/s| \ll 1$ and the field $\hat{\boldsymbol{\phi}}(i,\tau)$ slowly varying. We are then allowed to make an expansion up to quadratic order in $\boldsymbol{l}$, $\hat{\boldsymbol{\phi}}'$, and $\dot{\hat{\boldsymbol{\phi}}}$. The parametrization of the classical fields in (5) is different from that employed in Sénéchal's approach [24] to the two-leg ladder, and allows for generalization of the path-integral approach to ladders with an arbitrary number of legs. In particular, the fact that the field $\hat{\boldsymbol{\phi}}$ is chosen to depend on the site index i along the legs, but not on the index a labeling the sites along the rung, reflects the assumption that the correlation length be much greater than the total width of the ladder, i.e., $\xi \gg n_l a$, where ξ is the staggered spin-spin correlation length and a the lattice spacing. This is well supported by several numerical works (see, for instance, [28] and [29]). The constraint $\hat{\boldsymbol{\Omega}}_a^2(i) = 1$ implies $\hat{\boldsymbol{\phi}}^2(i) = 1$ and $\hat{\boldsymbol{\phi}}(i) \cdot \boldsymbol{l}_a(i) = 0$. As far as the intraleg term of $H(\tau)$ is concerned, everything proceeds as for the continuum limit of the one-dimensional Heisenberg chain (see [27] for a detailed calculation) and hence we write just the final result,

$$\sum_{a,i} J_a s^2 \hat{\boldsymbol{\Omega}}_a(i,\tau) \cdot \hat{\boldsymbol{\Omega}}_a(i+1,\tau) \approx \int dx \left[\frac{(s^2 \sum_a J_a)}{2} \hat{\boldsymbol{\phi}}'^2(x,\tau) + 2\sum_a J_a |\boldsymbol{l}_a(x,\tau)|^2 \right]. \qquad (6)$$

For the interleg term we have instead

$$\hat{\boldsymbol{\Omega}}_a(i,\tau) \cdot \hat{\boldsymbol{\Omega}}_{a+1}(i,\tau) \approx -1 + \frac{|\boldsymbol{l}_a(i,\tau)|^2}{2s^2} + \frac{|\boldsymbol{l}_{a+1}(i,\tau)|^2}{2s^2} + \frac{\boldsymbol{l}_a(i,\tau) \cdot \boldsymbol{l}_{a+1}(i,\tau)}{s^2}. \qquad (7)$$

The term $H(\tau)$ in the action has then the continuum limit

$$H(\tau) = \frac{1}{2} \int dx \left[\left(s^2 \sum_a J_a \right) \hat{\boldsymbol{\phi}}'^2(x,\tau) + \sum_{a,b} \boldsymbol{l}_a(x,\tau) L_{a,b} \boldsymbol{l}_b(x,\tau) \right],$$

where $L_{a,b}$ is the same matrix defined in Ref. [26], i.e.,

$$L_{a,b} = \begin{cases} 4J_a + J'_{a,a+1} + J'_{a,a-1}, & a = b, \\ J'_{a,b}, & |a-b| = 1. \end{cases} \qquad (8)$$

($J'_{a,a-1} \equiv J'_{a-1,a}$ and $J'_{1,0} = J'_{n_l,n_{l+1}} = 0$ in the formula above). We have finally to evaluate the Berry phase term. We now need the formula for the variation of the Berry phase $\omega[\hat{\boldsymbol{\phi}}]$ upon a small change $\delta\hat{\boldsymbol{\phi}}$ [27]:

$$\delta\omega = \int_0^\beta d\tau \delta\hat{\boldsymbol{\phi}} \cdot (\hat{\boldsymbol{\phi}} \times \dot{\hat{\boldsymbol{\phi}}}). \qquad (9)$$

We have therefore, at leading order,

$$\sum_{i,a} \omega[\hat{\boldsymbol{\Omega}}_a(i,\tau)] = s \sum_{i,a} (-1)^{a+i} \omega[\hat{\boldsymbol{\phi}}(i,\tau)] + \sum_{a,i} \int_0^\beta d\tau [\hat{\boldsymbol{\phi}}(i,\tau) \times \dot{\hat{\boldsymbol{\phi}}}(i,\tau)] \cdot \boldsymbol{l}_a(i,\tau)$$
$$\equiv \Gamma[\hat{\boldsymbol{\phi}}] + \sum_{a,i} \int_0^\beta d\tau [\hat{\boldsymbol{\phi}}(i,\tau) \times \dot{\hat{\boldsymbol{\phi}}}(i,\tau)] \cdot \boldsymbol{l}_a(i,\tau). \qquad (10)$$

The first term in the last equation is the topological term. We shall return to it in a while, after having integrated out the fluctuation field $\boldsymbol{l}_a(i,\tau)$:

$$\int D[\boldsymbol{l}]\exp\int_0^\beta d\tau\int dx\left\{-\frac{1}{2}\sum_{a,b}\boldsymbol{l}_a(x,\tau)L_{a,b}\boldsymbol{l}_b(x,\tau)+i\sum_a[\hat{\boldsymbol{\phi}}(x,\tau)\times\dot{\hat{\boldsymbol{\phi}}}(x,\tau)]\cdot\boldsymbol{l}_a(x,\tau)\right\}=$$
$$\exp\left[-\int_0^\beta d\tau\int dx\frac{\sum_{a,b}L^{-1}_{a,b}}{2}|\hat{\boldsymbol{\phi}}\times\dot{\hat{\boldsymbol{\phi}}}|^2\right]=\exp\left[-\int_0^\beta d\tau\int dx\frac{\sum_{a,b}L^{-1}_{a,b}}{2}|\dot{\hat{\boldsymbol{\phi}}}|^2\right]. \qquad (11)$$

The integration over the field $\boldsymbol{l}_a(i)$ gives therefore the kinetic term of the NLσM. For the topological term

$$\Gamma[\hat{\boldsymbol{\phi}}]=s\sum_a(-1)^a\sum_i(-1)^i\omega[\hat{\boldsymbol{\phi}}(i,\tau)], \qquad (12)$$

we can simply observe that, since the field $\hat{\boldsymbol{\phi}}(i,\tau)$ does not depend on a, one has just the same topological term as for the chain times a factor which is 0 for even n_l and 1 for odd n_l,

$$\Gamma[\hat{\boldsymbol{\phi}}]=\begin{cases}\frac{\theta}{4\pi}\int_0^\beta d\tau\int dx\,\hat{\boldsymbol{\phi}}\cdot(\dot{\hat{\boldsymbol{\phi}}}\times\hat{\boldsymbol{\phi}}'), & n_l \text{ odd},\\ 0, & n_l \text{ even}.\end{cases} \qquad (13)$$

($\theta=2\pi s$ in the last equation.) In the case of the 2D (infinite) lattice, cancellation of the topological term results, as is well known [8–12], from taking the continuum limit also along the direction of the rungs, and holds at least for smooth field configurations. Here it is instead a direct consequence of the assumption that $\xi\gg n_l a$, which lies at the heart of the parametrization chosen in Eq. (5). The staggering of the field $\hat{\phi}$ along the rungs leads then to the result that the topological term survives only for odd-leg ladders, being of course significant only for half-odd integer spins. Putting everything together we found that the antiferromagnetic Heisenberg ladder system is mapped onto a (1 + 1) NLσM with the Euclidean Lagrangian

$$\mathcal{L}=\begin{cases}\frac{1}{2g}\left(\frac{1}{v_s}\dot{\hat{\boldsymbol{\phi}}}^2+v_s\hat{\boldsymbol{\phi}}'^2\right)\\ \quad+\frac{i\theta}{4\pi}\hat{\boldsymbol{\phi}}\cdot(\dot{\hat{\boldsymbol{\phi}}}\times\hat{\boldsymbol{\phi}}'), & n_l \text{ odd},\\ \frac{1}{2g}\left(\frac{1}{v_s}\dot{\hat{\boldsymbol{\phi}}}^2+v_s\hat{\boldsymbol{\phi}}'^2\right), & n_l \text{ even},\end{cases} \qquad (14)$$

where the NLσM parameters are defined by

$$g^{-1}=s\left(\sum_{a,b,c}J_aL^{-1}_{b,c}\right)^{1/2}, \qquad (15)$$

$$v_s=s\left(\frac{\sum_a J_a}{\sum_{b,c}L^{-1}_{b,c}}\right)^{1/2}. \qquad (16)$$

Let us remark that the NLσM velocity we have obtained coincides with the spin-wave velocity (see [26] for the calculation of the spin-wave velocity in our model Hamiltonian). In the particular case $n_l=2$ we obtain

$$g=\frac{1}{s}(1+J'/2J)^{1/2}, \qquad (17)$$

$$v_s=2sJ(1+J'/2J)^{1/2}. \qquad (18)$$

These values of the parameters coincide with the ones obtained by Sénéchal in [24] for the two-leg ladder. They are different, however, from the parameters obtained by Sierra through his Hamiltonian mapping to the NLσM [26]. He has for the n_l-leg ladder:

$$g^{-1}=s\left[2\sum_{a,b,c}J_aL^{-1}_{b,c}-\frac{1}{4}\delta_{n_l}\right]^{1/2}, \qquad (19)$$

$$v_s=s\left[2\frac{\sum_a J_a}{\sum_{b,c}L^{-1}_{b,c}}-\delta_{n_l}\frac{1}{(2\sum_{b,c}L^{-1}_{b,c})^2}\right]^{1/2}, \qquad (20)$$

where δ_{n_l} is equal to 1 for odd n_l and 0 for even n_l. For even n_l his g is smaller, while v_s is greater, by a factor of $\sqrt{2}$ as compared with ours. The two results coincide (and coincide with Affleck's as well, as they should) in the limit of a single chain, i.e., for $n_l=1$. They are again different for odd $n_l>1$. We argue that this is due to the introduction, in Ref. [26], of additional massive fields. The latter are decoupled from the σ-model field and are ultimately neglected in the effective theory developed in [26]. Nonetheless, they appear to affect the actual values of the parameters of the effective σ model. The agreement of our approach and of that of Ref. [26] for $n_l=1$ (when there is no room for massive fields) seems to give support to our conjecture.

In any case the general picture for the behavior of the spin ladder is the same as in [26]. The ladders with an even number of legs are mapped to a (1 + 1) NLσM without topological term and are therefore gapped, while the ones with an odd number of legs are described by a 1 + 1 NLσM with the topological term, and are gapless [7] for half-odd integer spin. In the case $J'_{a,a+1}=J'$, $J_a=J$, and $J'\ll J$ we have $g\sim 2/sn_l$; the coupling constant g gets therefore smaller and smaller when the number of legs increases and the NLσM enters in a weak coupling regime. In this regime we may use the formula $\Delta\sim\exp(-2\pi/g)$ [30] to estimate the spin gap for even-leg ladders. We obtain then $\Delta\sim\exp(-\pi sn_l)$; the gap decreases therefore with the number of legs, as observed in numerical simulations [28,29] and as expected from the fact that in the limit $n_l\longrightarrow\infty$ the difference between ladders with even and odd number of legs must disappear. Let us also remark that the ratio $\xi/n_la\sim\exp(\pi sn_l)/n_l$; the condition $\xi\ll n_la$ we supposed at the beginning of our calculation seems therefore to be satisfied (in a self-consistent way) better and better when the number of legs increases (at least for $J'\ll J$). In the opposite regime $J' \gg J$ the coupling constant becomes strong since $g \sim (J'/J)^{1/2}$ and we can therefore estimate the spin gap as $\Delta = v_s g$ in the strong coupling regime [7]. For two legs we get $\Delta \sim J'$ when $J' \gg J$, a result which also agrees with what was found in the literature through other techniques [18].

Note added.—Just after completion of this work a preprint by G. Sierra has appeared [31] in which the mapping of spin ladders to the NLσM is reviewed also within a path-integral formalism. The values for the NLσM parameters found by Sierra agree with ours.

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