Chiral Metal as a Heisenberg Ferromagnet

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The two-dimensional surface of an integer quantum hall multilayer is mapped onto a Heisenberg spin chain with ferromagnetic coupling. Using this mapping it is shown nonperturbatively that the surface states constitute a very anisotropic metal in the infinite size limit. For multilayers of finite size, two diffusive mesoscopic regimes are identified and the conductance fluctuations are calculated perturbatively for both. The Heisenberg spin-chain representation is used to study the directed wave problem and the exact result is obtained that the mean-square deflection of a directed wave grows as the square root of the propagation distance. [S0031-9007(97)02802-0]

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Dirty electronic systems exhibit a variety of phases many of which are not well understood [1]. For example, electrons moving in two dimensions under the influence of a magnetic field exhibit a sequence of localization-delocalization transitions as the magnetic field or some other system parameter is varied. Understanding of these transitions, which underlie the quantum hall effect, is based largely on numerical simulation [2]. The purpose of this Letter is to study the two dimensional (2D) electronic states that live on the surface of an integer quantum hall multilayer (sometimes called the bulk quantum hall effect). At the edge of each quantum hall layer the electrons circulate in one sense only and may be modeled as noninteracting chiral fermions [3]. If the layers are coupled by tunneling, the surface states comprise a 2D chiral electronic system (see Fig. 1). The Bechgaard salts are a natural realization [4], and it is also possible to fabricate an appropriate semiconductor heterostructure [5].

The key question from the point of view of quantum transport is whether the electronic wave functions are localized or extended along the direction of the field (the z direction in Fig. 1). This determines whether the system is metallic or insulating from the point of view of transport along the z direction. In either case, to fully characterize the transport it is not sufficient to study the disorder-averaged conductance: Localized electronic systems generally possess a very broad distribution of conductances. Although in contrast metallic systems of ordinary (rather than chiral) electrons do not have a broad conductance distribution, the conductance fluctuations of finite-sized or *mesoscopic* metallic grains have remarkable *universal* properties [6] (for example, the typical fluctuations are of order e^2/h—independent of the mean conductance or other details of the sample).

Quantum transport in the chiral model has previously been studied numerically [7] and by mapping onto field theories [8–10]. In particular, Balents *et al.* [10] have mapped the system onto a one-dimensional (1D) supersymmetric ferromagnetic spin chain using the well-established supersymmetric technique for performing disorder averages [11]. Using this mapping they are able to establish the important result that the 2D chiral model is metallic (in the sense that the conductance scales ohmically) even for arbitrarily strong disorder. This should be contrasted with conventional electronic systems in which metallic scaling is associated with large conductance and which are generically localized in two dimensions [1]. The surface states of the bulk quantum hall system are thus revealed to be a novel metallic phase with interesting localization and mesoscopic fluctuation physics that awaits exploration.

The chiral model [Eq. (1) below] that describes the surface of a quantum hall multilayer is of interest from another quite distinct point of view. It should apply whenever waves propagate in a medium that is sufficiently anisotropic to warrant neglect of backscattering in one direction. The problem of waves propagating in an anisotropic medium has been the focus of much attention and is known as the directed wave problem (see, for example, Ref. [12,13] and references therein). Although directed waves are described by the chiral model, the question one asks in this context is very different: The electron is assumed to be localized at a single point initially. Thereafter it moves ballistically in the chiral x direction and the wave function spreads (presumably

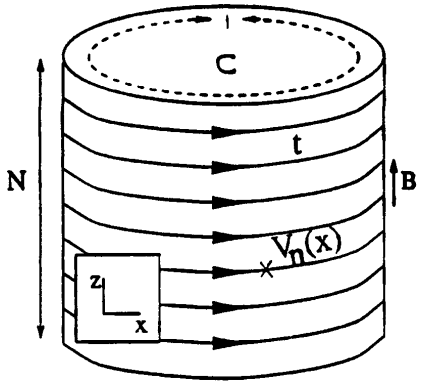

FIG. 1. A quantum hall multilayer.

diffusively) in the transverse z direction. The interesting questions here concern the growth of the wave-packet width and the fluctuations in the position of the center of the wave packet (denoted $[\langle n\rangle^2]_{\rm imp}$; the notation is explained below). The broadening of the width can be easily calculated, and the answer has been known since the 1970s [14]. $[\langle n\rangle^2]_{\rm imp}$ is more difficult to calculate as the average of four Green's functions is now required; it has not been evaluated within this model previously.

In this paper a new approach to disorder averaging is introduced which is distinct from the conventional replica or supersymmetry methods and is especially adapted to this system. Using this method it is possible to map the chiral model (in the limit of infinite size) onto a much simpler soluble 1D model: an ordinary Heisenberg ferromagnet. The absence of localization for the chiral system established by Refs. [7,8,10] then emerges nonperturbatively as a consequence of the well-known quadratic dispersion of ferromagnetic magnons. The advantages of this mapping onto an ordinary ferromagnet become evident when a more difficult calculation is attempted. For example, $[\langle n\rangle^2]_{\rm imp}$ can be expressed in terms of the matrix elements of two-magnon states of the ferromagnetic representation. Although magnons interact, it is not difficult to obtain the two-magnon eigenstates of a ferromagnet [15]. Carrying out such a calculation leads to the *exact* result that $[\langle n\rangle^2]_{\rm imp}$ grows as $x^{1/2}$. This exact result agrees with the numerical simulations of [16] but contradicts those of [17]; it also agrees with results obtained from a simplified lattice model of directed wave propagation introduced by Saul, Kardar, and Read [13,18,19]. In their model special assumptions are made about the disorder which make it possible to directly evolve the probability function (the modulus square of the wave function) without reference to the wave function itself. The results of this paper show that the same behavior results for a more generic disorder distribution.

Finally the properties of mesoscopic quantum hall multilayers are studied. Because of the anisotropy of the surface states *two* diffusive mesoscopic regimes can be identified. Following the terminology of Ref. [10] in the 1D diffusive regime electrons are typically able to wind around the sample in the chiral direction many times before diffusing across in the z direction. The opposite limit of sufficiently large circumference that winding paths are rare is called the zero-dimensional (0D) regime. The mesoscopic regimes are difficult to analyze due to the winding paths. Spectral correlations were computed in Ref. [10] in the 0D limit; however, spectral correlations are difficult to probe experimentally. Here conductance fluctuations are calculated in the two mesoscopic regimes using the standard methods of impurity averaged perturbation theory [6]. Universal conductance fluctuations of order e^2/h are found in the 1D limit coinciding with the result for ordinary metals. Interestingly, there is a crossover to much larger fluctuations in the 0D diffusive limit.

The precise model used for the multilayer surface is now described. For simplicity we shall focus on the case of just one filled Landau level. The surface of the multilayer is then a 2D chiral electronic system governed by the Schrödinger equation

$$\left(-iv\,\frac{\partial}{\partial x} + V_n(x) - E\right)G_E^R(n,x;n',x') - t\{G_E^R(n+1,x;n',x') + G_E^R(n-1,x;n',x')\} = -iv\,\delta_{n,n'}\delta(x-x'). \quad (1)$$

Here G_E^R is the retarded Green's function at frequency E. t produces interlayer hopping, and V_n is the disorder potential. Units are chosen so that the edge velocity $v = 1$, the interlayer separation $a = 1$, and $\hbar = 1$. The anisotropy and chiral character of the model are reflected by the fact that the equation is first order in the chiral x direction, whereas it is second order in the transverse direction.

A mapping onto a 1D problem is obtained by noting that in the limit of infinite size (but in that limit only) the circumference $C \to \infty$ and the retarded Green's function obeys the chiral boundary condition $G_E^R(n,x;n',x') = 0$ for $x < x'$. Because of this boundary condition, it is possible to interpret Eq. (1) as the *time-dependent* Schrödinger equation for a 1D tight-binding model with the chiral coordinate x identified as time and $G_E^R(n,x,;n',x')$ identified as the time-domain retarded Green's function. Note that the on-site energies of the tight-binding model fluctuate in time. It is convenient to make a gauge transformation $G^R \to G^R \exp[i\gamma_n(x) - i\gamma_{n'}(x')]$ where $\partial\gamma_n(x)/\partial x = V_n(x) - E$. In this gauge Eq. (1) becomes

$$-i\,\frac{\partial}{\partial x}\,G_E^R(n,x;n',x') - t_n(x)G_E^R(n+1,x;n',x') - t_{n-1}^*(x)G_E^R(n-1,x;n',x') = -i\,\delta_{n,n'}\delta(x-x'). \quad (2)$$

Here $t_n(x) \equiv t\exp i[\gamma_{n+1}(x) - \gamma_n(x)]$.

The tight-binding model can be rewritten in second-quantized language by introducing $c_n^{R\dagger}$ and c_n^R which create and annihilate fermions on site n of the tight-binding lattice and which evolve in the chiral time direction according to the Hamiltonian

$$h_{1\rm d}^R(x) = \sum_n \{t_n(x)c_n^{R\dagger}c_{n+1}^R + t_{n-1}^*(x)c_n^{R\dagger}c_{n-1}^R\}. \quad (3)$$

In this language the Green's function is given by

$$G^R(n,x;n',x') = \langle 0|c_n^R P\exp\left(i\int_{x'}^{x} dx_1\, h_{1\rm d}(x_1)\right)c_{n'}^{R\dagger}|0\rangle \quad (4)$$

for $x > x'$. The symbol $P\exp(\,)$ denotes a time-ordered exponential. Note that there is no vacuum amplitude in the denominator because we are calculating the single particle Green's function rather than the propagator for a single particle added to a filled Fermi sea, which is the object usually studied in many-body physics [20]. The absence of a denominator in Eq. (4) is a crucial simplification that allows disorder averaging as discussed below.

The complex conjugate of the Green's function can be calculated from an expression analogous to Eq. (4) by introducing conjugate fermions, $c^{A\dagger}$ and c^A, which evolve according to a conjugate Hamiltonian [obtained from Eq. (3) by making the replacements $c^R \to c^A$, $t \to -t^*$, and $t^* \to -t$].

Localization, or its absence, is established by calculation of the disorder-averaged diffuson propagator $|G_E^R(n,x;n',x')|^2$. For this purpose it is neccessary to simultaneously introduce both sets of fermions evolving according to the total Hamiltonian

$$h_{1d}(x) = h_{1d}^R + h_{1d}^A = \sum_n \{t_n A_n + t_n^* A_n^\dagger\}. \tag{5}$$

Here $A_n = c_n^{R\dagger} c_{n+1}^R - c_{n+1}^{A\dagger} c_n^A$. The diffuson is then given by

$$|G(n,x;n',x')|^2 = \langle 0|c_n^A c_n^R P\exp\left(i\int_{x'}^{x} dx_1\, h_{1d}(x_1)\right) \times c_{n'}^{R\dagger} c_{n'}^{A\dagger}|0\rangle. \tag{6}$$

The task now is to average over different realizations of the disorder potential which is taken to be Gaussian white noise with correlations $[t_n^*(x)t_m(x')]_{\rm imp} = D\delta_{n,m}\delta(x - x')$ and $[t_n(x)]_{\rm imp} = [t_n(x)t_m(x')]_{\rm imp} = 0$ [8–10], where $[\ldots]_{\rm imp}$ denotes an average over disorder. Because of the absence of a denominator in Eq. (6) this average is easily performed. It is neccessary only to evaluate

$$\left[P\exp\left(i\int_{x'}^{x} dx_1\, h_{1d}(x_1)\right)\right]_{\rm imp} = \exp\{-\overline{h}_{\rm int}(x - x')\}, \tag{7}$$

where

$$\overline{h}_{\rm int} = \frac{1}{2} D \sum_n (A_n^\dagger A_n + A_n A_n^\dagger). \tag{8}$$

Equation (8) can be verified by expanding the exponentials in Eq. (7). The rough content of Eqs. (7) and (8) is that, for calculating averages, the fermions may be taken to evolve according to an effective Hamiltonian $\overline{h}_{\rm int}$ which is not random and does not depend on x. It is an interacting Hamiltonian since A_n is bilinear.

Consider a state in which a single site n is simultaneously occupied by R and A fermions—below this state will be identified as a magnon localized at site n. The effect of $\overline{h}_{\rm int}$ on such a state is to cause both fermions to hop together onto a neighboring site. This physics can be brought out clearly by defining $J_n^z \equiv \frac{1}{2}(c_n^{R\dagger} c_n^R - c_n^A c_n^{A\dagger})$, $J_n^+ \equiv J_n^x + iJ_n^y \equiv c_n^{R\dagger} c_n^{A\dagger}$, $J_n^- \equiv (J_n^+)^\dagger$, which satisfy the su(2) algebra, and $N_n \equiv c_n^{R\dagger} c_n^R + c_n^A c_n^{A\dagger}$, which commutes with all the J's. In terms of these operators

$$\overline{h}_{\rm int} = D\sum_n \left(N_n - \frac{1}{2} N_n N_{n+1} - 2\vec{J}_n \vec{J}_{n+1}\right) \tag{9}$$

—evidently a Heisenberg ferromagnet. As usual the vacuum $|0\rangle$ is the ground state and exact low-lying excitations are magnons obtained by constructing plane waves from the localized magnons mentioned above. Explicitly, a magnon of wave vector k is given by $\sum_n J_n^+ \exp ikn|0\rangle$ and has eigenvalue $2D(1 - \cos k)$.

The exact diffuson propagator can now be straightforwardly calculated by substituting Eq. (7) in Eq. (6) and expanding $c_{n'}^{R\dagger} c_{n'}^{A\dagger}|0\rangle$ and $\langle 0|c_n^A c_n^R$ in terms of magnons. The result is

$$[|G(n,x;n',x')|^2]_{\rm imp} = \theta(x - x') \times \int_{-\pi}^{+\pi} \frac{dk}{2\pi} \exp ik(n - n') \exp\{-2D(1 - \cos k)(x - x')\} \tag{10}$$

in agreement with Eq. (98) of Balents *et al.* [10]. The physical content of Eq. (10) is that the electrons move ballistically in the chiral direction and diffuse in the transverse direction (diffusion constant $= D$) [21].

Next consider the directed wave problem. The electron is assumed to be initially localized at the origin of the coordinate system $n' = 0, x' = 0$. After it moves ballistically in the chiral direction to a location x, the amplitude to be in layer n is given by $G^R(n,x;n' = 0,x' = 0)$ and the position of the wave-packet center $\langle n\rangle = \sum_n n|G^R(n,x;0,0)|^2$. By symmetry, evidently $[\langle n\rangle]_{\rm imp} = 0$; and the mean-square deflection of the wave-packet center is therefore

$$[\langle n\rangle^2]_{\rm imp} = \sum_{n,m} nm[|G(n,x;0,0)|^2|G(m,x;0,0)|^2]_{\rm imp}. \tag{11}$$

The large x asymptotic behavior of $[\langle n\rangle^2]_{\rm imp}$ is desired.

To perform this calculation it is neccessary to introduce *two* sets each of R and A fermions (c^R, c^A, d^R, d^A) which evolve according to the Hamiltonian of Eq. (5) but with $A_n \to c_n^{R\dagger} c_{n+1}^R - c_{n+1}^{A\dagger} c_n^A + (c \to d)$. Repeating the previous arguments,

$$[|G(n,x;0,0)|^2|G(m,x;0,0)|^2]_{\rm imp} = \langle 0|c_n^A c_n^R d_m^A d_m^R \exp\{-\overline{h}_{\rm int} x\} d_0^{R\dagger} d_0^{A\dagger} c_0^{R\dagger} c_0^{A\dagger}|0\rangle. \tag{12}$$

Here $\overline{h}_{\rm int}$ is given by Eq. (8) but with A_n redefined as above. The important states are (localized) two-magnon states which are of two kinds. In the first type a c^R-c^A pair occupies one site while a d^R-d^A pair occupies another; in the second type c^R is paired with d^A and c^A with d^R [22]. These states are closed under the action of $\overline{h}_{\rm int}$ which generally causes paired fermions to hop together to a neighboring site. An exception is when the two pairs occupy adjacent sites, in which case the fermions may switch

partners and a state of one kind is transformed into the other. Thus the two-magnon states define a sort of two-body problem with a contact interaction of a kind familiar from ordinary ferromagnetism [15]. Following the standard method explicit forms for the two-magnon eigenstates are obtained which can be used to straightforwardly evaluate Eqs. (11) and (12); details will be given elsewhere [19]. Contrary to the naive assumption that the center undergoes a random walk, and hence $[\langle n\rangle^2]_{\rm imp} \sim x$, exact calculation reveals that $[\langle n\rangle^2]_{\rm imp} = \sqrt{(2Dx)/\pi}$.

Finally consider multilayers that are of finite size in both chiral and transverse direction thus permitting electrons to wind around the sample in the chiral direction. During each circumnavigation, the electrons will typically diffuse a distance $\sqrt{DC}$ in the transverse direction. Consequently there are two distinct regimes depending on whether $N \ll \sqrt{DC}$ (0D limit) or $N \gg \sqrt{DC}$ (1D limit). Here N = number of layers in the multilayer. Finite-sized samples are difficult to analyze due to the complex interference produced by winding paths. However, the conductance fluctuations can be calculated using diagrammatic perturbation theory within the approximation normally used for diffusive electrons [6]. The result is $[(\delta g)^2]_{\rm imp} = A(e^2/h)^2$ (1D limit) and $[(\delta g)^2]_{\rm imp} = A'\{(CD)/N^2\}(e^2/h)^2$ (0D limit). Here A and A' are constants of order unity which we have not computed explicitly [23]. Conductance fluctuations are of order e^2/h in the 1D limit as they are for ordinary metallic grains. The larger nonuniversal result for the 0D limit may be interpreted as follows: The electron moves a distance $N^2/D \ll C$ in the chiral direction before it diffuses into the phase-randomizing probes. The sample therefore breaks up into CD/N^2 incoherent blocks, each with independent conductance fluctuations of order e^2/h. Although the perturbative results given here have an appealing physical interpretation, the validity of these results deserves further study via simulation or nonperturbative analysis (which may be possible in the 0D limit [24]). By the familiar "ergodic hypothesis" of mesoscopic physics [6] we expect the statistical fluctuations of the conductance calculated here would be experimentally manifested as fluctuations in the conductance of a given specimen when a tunable parameter, e.g., the magnetic field, is varied.

It is interesting to compare the method of this paper with other field theory representations of the chiral model. Because single particle properties are being calculated, Eqs. (3)–(8) would remain valid even if the fermions were replaced by bosons ($c^R, c^A \to b^R, b^A$) yielding an interacting boson representation of the chiral model rather than the Heisenberg ferromagnet analyzed here. If fermions *and* a redundant set of bosons are introduced, the supersymmetric representation of Balents *et al.* [10] results. It is very convenient to be rid of the unneeded bosons, e.g., while analyzing the directed wave problem; but it is important to emphasize that the bosons are optional only for the infinite system. For a finite circumference the boundary conditions on Eq. (1) change, and it becomes necessary to introduce bosons whether operator methods (this paper) or functional methods (Balents *et al.*) are used.

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