Scaling of the Mean Velocity Profile for Turbulent Pipe Flow

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An experimental investigation was conducted to determine the scaling of the mean velocity profile for a fully developed, smooth pipe flow. Measurements of the mean velocity profiles and static pressure gradients were performed at 26 different Reynolds numbers between 31×10^3 and 35×10^6. The profiles indicate two overlap regions: one which scales as a power law and one which scales as a log law, where the log law is only evident when the Reynolds number exceeds approximately 300×10^3. It is proposed that the velocity scales for the inner and outer regions are different, which is contrary to commonly accepted beliefs. [S0031-9007(96)02054-6]

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In this Letter we investigate the scaling of the mean velocity profile in fully developed, turbulent flow in a smooth pipe. Fully developed flow requires that the flow is free from entrance effects so that all flow properties become independent of streamwise position, and they depend only on the Reynolds number and surface roughness of the pipe wall. Here the Reynolds number Re is given by $\overline{U}D/\nu$, where $\overline{U}$ is the average velocity, D is the diameter of the pipe, and ν is the kinematic viscosity. When the roughness is small enough, the pipe is said to be smooth, and the flow depends only on the Reynolds number. Despite the large number of previous studies on this type of flow (note especially [1]), the existing data do not cover a very large range of Reynolds numbers, and questions of accuracy make it difficult to unambiguously establish the scaling of the mean velocity profile. An experiment was therefore performed to provide high-quality, mean-flow data over a range of Reynolds numbers from 31×10^3 to 35×10^6 in a fully developed, smooth pipe flow.

In fully developed, laminar pipe flow the mean velocity profile can be described by a single length and velocity scale, and the equations of motion yield a similarity solution for the velocity profile. In fully developed, turbulent pipe flow, however, a single similarity solution for the mean velocity profile has not been found or may not exist. Instead, the mean velocity profile in turbulent pipe flow is usually divided into two regions, a near-wall or inner region, and a core or outer region, and separate similarity solutions are sought for each region.

For the inner region, it can be argued that the viscosity and wall shear stress are the important parameters governing the velocity distribution [2]. That is,

$$U = f_0(y, u_\tau, \nu), \tag{1}$$

where y is the distance from the wall and u_τ is the velocity scale in the inner region. The inner velocity scale is known as the "friction" velocity and is defined as $u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\frac{D}{4\rho}\frac{dP}{dx}}$, where τ_w is the wall shear stress, ρ is the fluid density, and dP/dx is the mean streamwise pressure gradient, which is independent of streamwise position in a fully developed pipe flow. The length scale in the inner region is given by the quantity ν/u_τ.

In the outer region, it can be proposed that the wall acts to reduce the velocity below the maximum or centerline velocity in a manner which is independent of viscosity [2]. Here the velocity profile should behave as

$$U_{CL} - U = g_0(y, R, u_o), \tag{2}$$

where U_{CL} is the centerline velocity and u_o is the velocity scale in the outer region. The length scale in the outer region is the radius R.

Dimensional analysis of Eqs. (1) and (2) leads to

$$U^+ = f(y^+), \tag{3}$$

and

$$(U_{CL} - U)/u_o = g(\eta), \tag{4}$$

where $U^+ = U/u_\tau$, $y^+ = yu_\tau/\nu$, $\eta = y/R$, and f and g are the dimensionless forms of f_0 and g_0, respectively. Equation (3) is known as the *law of the wall*, and is valid only in the inner region. It can be shown from the equations of motion that f is linear very close to the wall ($y^+ < 5$), and we expect that Eq. (3) is valid further from the wall than the linear region but not into the outer region ($0 < y^+ \ll R^+ = Ru_\tau/\nu$). Equation (4) is known as the *defect law*, and is valid only in the outer region ($0 \ll \eta < 1$) where viscosity is not important.

It is conventionally argued that the velocity scale for the outer region is determined by the wall shear, so that $u_o = u_\tau$. Using this assumption, Millikan [3] proposed that at large enough Reynolds numbers there may be a region of overlap where both Eqs. (3) and (4) are simultaneously valid. In this overlap region, where $\nu/u_\tau \ll y \ll R$, a logarithmic mean velocity profile results. Since this is an overlap region, it can be expressed using inner or outer scaling variables. When nondimensionalized using inner scaling variables we obtain

$$U^+ = \kappa^{-1} \ln y^+ + B, \tag{5}$$

where κ is known as von Kármán's constant and B is a constant that depends on the inner limit of validity for the log law. Both constants are empirical. When nondimensionalized using outer scaling variables the resulting equation is

$$U_{CL}^{+} - U^{+} = -\kappa^{-1} \ln \eta + B^{*}, \qquad (6)$$

where B^* is an empirical constant that depends on the outer limit of validity for the log law. In a pipe flow, typical values for κ, B, and B^* are 0.41, 5.2, and 0.65, respectively, and the logarithmic region is believed to exist for $50\nu/u_\tau < y < 0.15R$.

A logarithmic overlap region with constants that are independent of the Reynolds number, is the "conventional wisdom" in pipe flow and indicates Reynolds number similarity. However, despite the large amount of data available for turbulent pipe flow, there is still some doubt as to the validity of the underlying scaling arguments embodied in Eqs. (5) and (6). For example, a power-law scaling for the mean velocity profile can be attained by both heuristic arguments (see [4], for example) and overlap arguments (see [5], for example).

To study these issues, an experimental apparatus was built to enable accurate measurements across a wide range of Reynolds numbers, up to large values. Compressed air was chosen as the working fluid to reduce costs. A closed-loop system was built with the test pipe located inside high-pressure piping (see Fig. 1). The test pipe had a nominal diameter of 129 mm. The primary test section was located 196 D downstream of the contraction and 6 D upstream of the exit diffuser. After honing, the inside of the test pipe was polished to an rms surface finish ($k/2$) of 0.15 ± 0.03 μm which corresponds to an average viscous height ($k^+ = ku_\tau/\nu$) of 2.7 ± 0.5 at a Reynolds number of 40×10^6. A surface is usually considered smooth when $k^+ < 5$ [6]. For further details of the experimental facility, see [7,8].

Measurements of the mean velocity profiles and static pressure gradients were performed at 26 different Reynolds numbers between 31×10^3 and 35×10^6. Each Reynolds number was achieved by varying either the density or flow rate. The flow was assumed to be incompressible since the maximum Mach number was less than 0.08 for all surveys. The temperature was always near ambient (295–300 K) and the pressure was varied between 1 and 186 atm. At these temperatures and pressures, air follows the ideal-gas relation to within $\pm 2.5\%$. Even so, the density and viscosity were calculated from the absolute pressure and temperature using real-gas relationships. For a description of these relationships and a comparison with experimental viscosity and density data on air, see [8]. The static pressure gradients were found using 20 wall taps (0.79 mm diameter) equally spaced over 25 D, in the region between the secondary and primary access ports (see Fig. 1). The velocity profiles were measured at the primary test section with a 0.90 mm diameter round Pitot probe. For each velocity profile, the outer surface of the Pitot probe was positioned within 0.04 mm of the wall, and was then traversed through 3/4 of the diameter. The overall uncertainty in the distance from the wall to the center of the Pitot probe was less than ± 0.05 mm. Pitot-probe corrections were applied according to [9] (see [8] for details). The worst case uncertainty for a differential pressure measurement was less than $\pm 0.40\%$ of the reading. The friction velocity and average velocity had an uncertainty of $\pm 0.45\%$ and $\pm 0.30\%$, respectively. The Reynolds number had an uncertainty of $\pm 0.93\%$, and the normalized velocity (U^+) had an uncertainty of $\pm 0.57\%$. The circumferential asymmetries in the flow were found to be negligible (see [8] for a description of the symmetry measurements).

In Fig. 2, we plot 13 mean velocity profiles normalized by inner scaling variables. Inspection of the individual velocity profiles reveals that for $y^+ < 0.1R^+$ ($y < 0.1R$), the mean velocity profile is independent of the Reynolds number, or equivalently, the outer length scale R. In Figs. 3 and 4 we plot all 26 velocity profiles normalized by inner scaling variables for $y^+ < 0.1R^+$. The results indicate that the mean velocity profile is independent of the Reynolds number and consists of two distinct regions: a power-law region for $50 < y^+ < 500$ or $0.1R^+$, and a log-law region for $500 < y^+ < 0.1R^+$.

In Fig. 3 the data are shown using log-log coordinates in order to emphasize the power-law dependence. The exponent is approximately 0.137 which is close to 1/7. The velocity data are within $\pm 0.78\%$ (95% confidence interval) of the power law shown. The existence of a power law implies that viscosity is still an important parameter for $y^+ < 500$. The viscous dependence suggests that this region is part of the inner region, but it could also be a separate overlap region in the mean velocity profile exhibiting incomplete similarity (that is, it depends on the Reynolds number). This is not the same type of

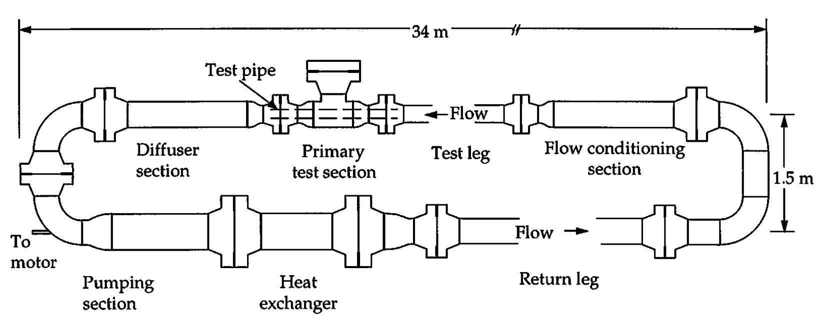

FIG. 1. The layout of the SuperPipe facility. The flow direction is counterclockwise.

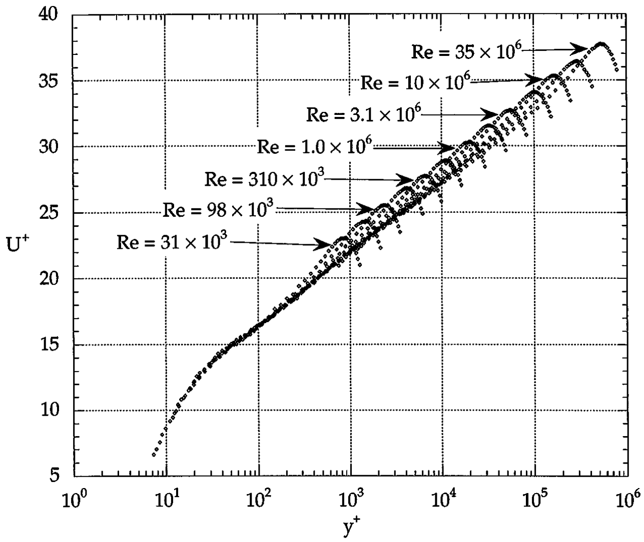

FIG. 2. Linear-log plot of velocity profiles normalized using inner scaling variables for 13 different Reynolds numbers between 31×10^3 and 35×10^6.

incomplete similarity suggested in [4] where the empirical constants in the power law depend on the Reynolds number and a log law is not obtained even at very large Reynolds numbers. Our results indicate that the power-law scaling exists in a discrete region between the inner region and outer region or logarithmic overlap region, depending on the magnitude of the Reynolds number, and the empirical constants in the power law do not depend on the Reynolds number when expressed using inner scaling variables. We argue that this region is not the overlap region expected at a very large Reynolds number, but an intermediate overlap region that covers the range of y^+ at which most previous experiments have been performed. At a very large Reynolds number, another overlap region is apparent, and the scaling in this region appears to be logarithmic.

In Fig. 4 the data are shown using linear-log coordinates in order to emphasize the log-law dependence. For $y^+ > 500$, a log law with a value of κ equal to 0.436 and an additive constant of 6.13 is in excellent agreement with the velocity data up to $y^+ \approx 0.1R^+$. The value of κ was found by curve fitting the friction factor data (see [7] or [8] for a complete description of this procedure). The constant B was found from a curve fit of the velocity profiles using $\kappa = 0.436$ in the range $500 < y^+ < 0.1R^+$. The first three data points near the wall were neglected due to the unacceptable uncertainty in their position, and only Reynolds numbers with at least ten data points in the assumed logarithmic region were analyzed. The individual values of B were averaged (no weighting) for each Reynolds number. This value of B shows no Reynolds number dependence and has an average value of 6.13 with a standard error of ± 0.04. The velocity data are within $\pm 0.68\%$ (95% confidence interval) of the log law shown.

If $y \approx 0.1R$ is the near-wall limit of the outer region, it would appear that a logarithmic overlap region does not exist for $R^+ < 5 \times 10^3$, which for a pipe corresponds to $\mathrm{Re} < 300 \times 10^3$. To distinguish a logarithmic overlap region over an order of magnitude in y^+ requires $R^+ > 5 \times 10^4$ ($\mathrm{Re} > 3 \times 10^6$ for a pipe) which has only been achieved here, and in the investigations by [1] and [10]. Therefore, the Reynolds number dependence of κ observed by many investigators may well be due to the fact that the scaling is not a log law, but appears to be a power law for $50 < y^+ < 500$. For instance, the value of κ given by $\kappa = (y^+ \partial U^+ / \partial y^+)^{-1}$ varies from 0.49 to 0.36 for $50 < y^+ < 500$ which is consistent with the variation observed by previous investigators (for example, [11]).

The power-law and log-law scaling in the mean velocity profile can be explained with an overlap argument: incomplete similarity in the power-law region and full similarity in the log-law region. The existence of both

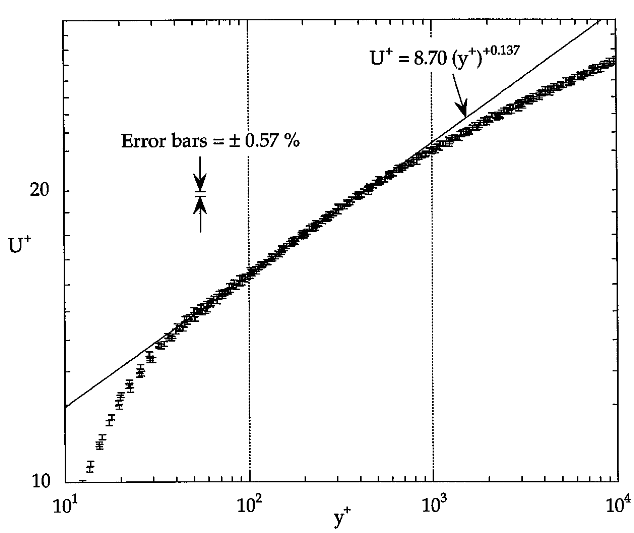

FIG. 3. Log-log plot of velocity profiles within $0.1R$ of the wall normalized using inner scaling variables for 26 different Reynolds numbers between 31×10^3 and 35×10^6.

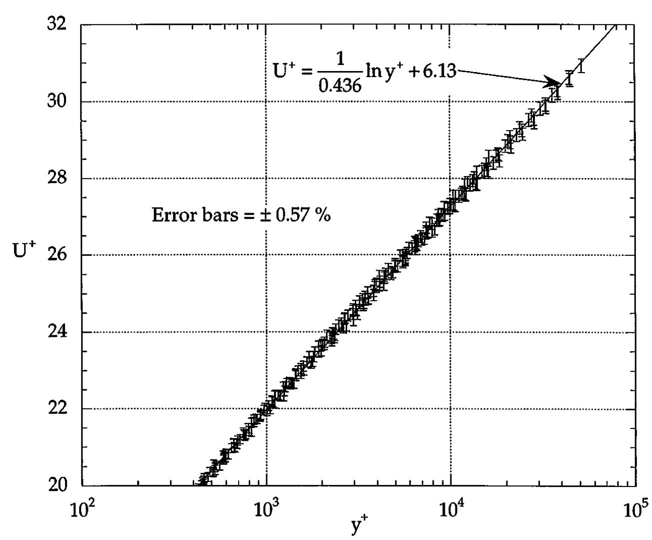

FIG. 4. Linear-log plot of velocity profiles within $0.1R$ of the wall normalized using inner scaling variables for 26 different Reynolds numbers between 31×10^3 and 35×10^6.

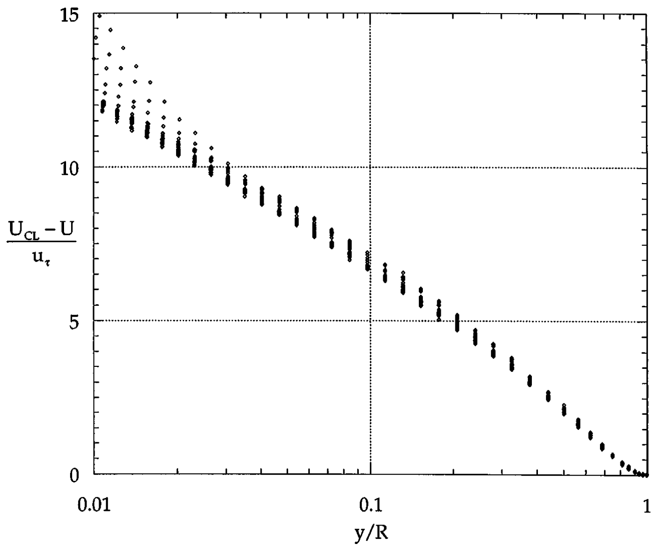

FIG. 5. Velocity profiles normalized using the conventional outer velocity scale for 26 different Reynolds numbers between 31×10^3 and 35×10^6.

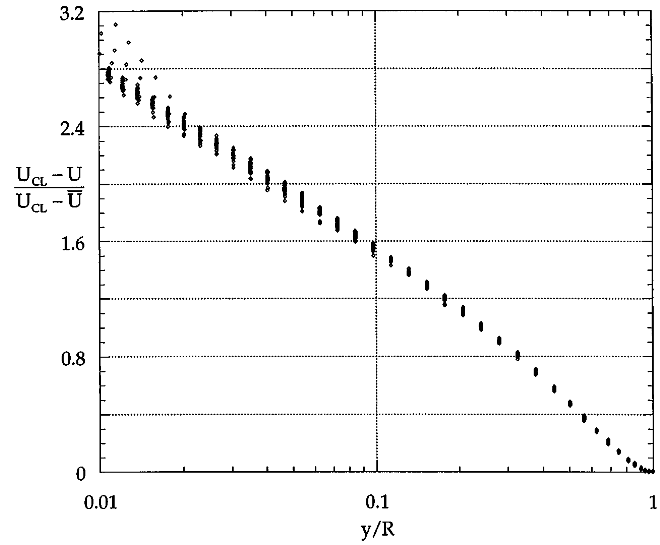

FIG. 6. Velocity profiles normalized using the proposed outer velocity scale for 26 different Reynolds numbers between 31×10^3 and 35×10^6.

regions at sufficiently high Reynolds numbers is consistent with a proposal which includes a new velocity scale, other than u_τ, for the outer region [u_o in Eq. (4)]. At high Reynolds numbers, this outer velocity scale must be proportional to the inner velocity scale u_τ for this proposal to yield a log law. An outer velocity scale which exhibits the proper behavior is given by $u_o = U_{CL} - \bar{U}$. At low Reynolds numbers, u_o/u_τ is a nonlinear function of the Reynolds number, but at high Reynolds numbers, u_o/u_τ is independent of the Reynolds number (i.e., $u_o/u_\tau = \text{const}$). The proposed velocity scale was used to normalize the velocity profiles in the outer region ($0.1R < y < R$). Inspection of Figs. 5 and 6 indicates that the velocity profiles normalized by $U_{CL} - \bar{U}$ are in significantly better agreement than the velocity profiles normalized by u_τ. We believe that $U_{CL} - \bar{U}$ is the correct velocity scale in the outer region of a pipe and should be used instead of u_τ. At sufficiently high Reynolds numbers ($R^+ > 5 \times 10^3$ or $500\nu/u_\tau < y < 0.1R$), the scaling in this overlap region was found to be logarithmic and $U_{CL} - \bar{U}$ is proportional to u_τ. Using $U_{CL} - \bar{U}$ as the velocity scale, the log law can be accurately represented by the equation

$$\frac{U_{CL} - U}{U_{CL} - \bar{U}} = \frac{1}{1.89} \ln \eta + 0.348 . \qquad (7)$$

The new outer velocity scale presented here was established for a pipe flow. For similar values of R^+, we may expect channel flow and boundary layers to scale the same way as pipe flow. An equivalent outer velocity scale for a boundary layer is given by $u_o = U_e \delta^*/\delta$, where U_e is the freestream velocity, δ^* is the displacement thickness, and δ is the boundary-layer thickness. At high Reynolds numbers, δ^*/δ should be proportional to $\sqrt{C_f}$ (C_f is known as the skin friction coefficient) for a logarithmic overlap region to exist. For a boundary layer some controversy exists over whether the scaling in the overlap region is a log law or a power law [5]. To observe a log law over an order of magnitude in y^+ would require a δ^+ of 50×10^3 which is rarely measured in laboratory experiments [12]. Therefore, for most boundary-layer experiments, the scaling in the overlap region may be a power law, although it should be pointed out that the scaling proposed by [5] (which is based on U_e) is not the same as the scaling proposed here.

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[1] J. Nikuradse, Forsch. Arb. Ing.-Wes. No. 356, 1932, English translation NACA TT F-10, 359.
[2] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1972).
[3] C. B. A. Millikan, in *Proceedings of the Fifth International Congress of Applied Mechanics* (Wiley, New York, 1938), pp. 386–392.
[4] G. I. Barenblatt, J. Fluid Mech. **248**, 513 (1993).
[5] W. K. George, P. Knecht, and L. Castillo, in *Proceedings of the Thirteenth Biennial Symposium on Turbulence* (University of Missouri-Rolla, Rolla, 1992).
[6] H. Schlichting, *Boundary-Layer Theory* (McGraw-Hill, New York, 1979), 7th ed.
[7] M. V. Zagarola, A. J. Smits, S. A. Orszag, and V. Yakhot, AIAA Report No. 96-0654, 1996.
[8] M. V. Zagarola, Ph.D. thesis, Princeton University, 1996.
[9] S. H. Chu, Prog. Aerosp. Sci. **16**, 147–223 (1975).
[10] J. Dickinson, Laval University internal report, 1975.
[11] J. O. Hinze, in *The Mechanics of Turbulence, International Symposium of the National Scientific Research Center, Marseille* (Gordon and Beach, New York, 1961).
[12] A. J. Smits and J. P. Dussauge, *Turbulent Shear Layers in Supersonic Flow* (AIP Press, New York, 1996).