Nonlinear Landau Damping in Collisionless Plasma and Inviscid Fluid

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The long-time nonlinear evolution of generic initial perturbations in stable Vlasov plasma and two-dimensional (2D) ideal fluid is studied. Even without dissipation, these systems relax to new steady states (Landau damping). The asymptotic damping laws are found to be algebraic, such as t^{-1} for 1D plasma potential, or $t^{-5/2}$ for evolving stream function in a flow with nonvanishing shear. The rate of the relaxation is fast so that phase-space/fluid-element displacement in certain directions is uniformly small, implying that decaying Vlasov and 2D fluid turbulences are not ergodic. [S0031-9007(97)02778-6]

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Part of the challenge facing the theory of turbulence is that it is extremely difficult to make exact statements about the long-time behavior of a nonintegrable system that go beyond the mere consequences of applicable conservation laws. For chaotic systems with a few degrees of freedom, there are a few results like this, including the little-known Sundman's theorems for the three-body problem (cf. Ref. [1], pp. 49–68) and the famous Kolmogorov-Arnold-Moser theory [2]. Here an attempt is made to draw certain long-time conclusions about the nonlinear evolution in a Vlasov plasma and in a 2D ideal fluid. We study the dynamics of the relaxation of a generic initial perturbation in these systems and derive algebraic damping laws for the perturbation. As in the above finite-dimensional examples, our continuous findings imply the lack of ergodicity, with grave implications for several statistical theories of turbulence.

We start with the 1D Vlasov-Poisson system for the electron distribution function $f(x,v,t) = f_0(v) + \tilde{f}(x,v,t)$ and the electric field $E(x,t) = -\partial_x \phi(x,t)$,

$$(\partial_t + v\partial_x + E\partial_v) f = 0,$$

$$\partial_x E = \int_{-\infty}^{\infty} f\, dv - 1\,, \qquad (1)$$

describing nonlinear plasma waves on a uniform ion background. In Eq. (1), the time t is normalized to the inverse plasma frequency ω_{pe}^{-1}, and x is measured in Debye lengths $r_D = v_e/\omega_{pe}$, where v_e is the electron thermal velocity, the unit for v. The problem has two basic dimensionless parameters, the nonlinearity $\epsilon \sim \tilde{f}/f_0$ and the typical wave number kr_D of the initial perturbation.

The original solution of the initial-value problem for Vlasov plasma by Landau [3] is strictly linear, meaning that ϵ is the smallest parameter of the problem. We will not assume either of the parameters ϵ or k small or large; instead, the largest, or the only large, parameter in our treatment will be time. The long-time limit is intrinsically nonlinear, because the linearization of the Vlasov-Poisson system fails for t larger than the particle bounce time $\tau_b \simeq \epsilon^{-1/2}$ [4]. This happens because the fluctuations of the distribution function do not decay, but rather develop free-streaming-type small scales, $\tilde{f}(x,v,t) \simeq \tilde{f}(x - vt, v, 0) \sim \epsilon$, and the nonlinearity, $\partial_v \tilde{f}/f_0'(v) \sim \epsilon t$, increases secularly with time.

The previous analytical work on the nonlinear Vlasov plasma includes the exact special stationary solutions of Bernstein, Greene, and Kruskal (BGK) [5] and the nonstationary theory of O'Neil for $\tau_b \leq t \ll \epsilon^{-1}$ and small k [4]. O'Neil showed that the damping rate of the wave, $\gamma(t) = \dot{\phi}/\phi$, starts oscillating about zero on the trapping time scale τ_b with a decreasing amplitude. The analytical theory of the trapping oscillations is possible because of the unchanged shape of the wave for $t \ll \epsilon^{-1}$. The currently prevailing conjecture is that nonlinear plasma waves, after several trapping oscillations, settle to a stationary stable BGK wave. This conclusion appears to be backed by numerical simulation [6], although numerical evidence is inconclusive for the long-time limit. More importantly, the stability of the nonlinear BGK waves remains an outstanding issue. This author is not aware of any single example of a stable BGK wave; moreover, all analytically written BGK waves appear linearly unstable [6], and the only known nonlinear stability criterion [7], $df_0/dH < 0$, where $H(x,v) = \phi(x) + v^2/2$ is the particle energy, holds for no periodic BGK wave ([8], p. 85). This suggests that the Landau damping will not be arrested by nonlinearity; however, the nature of the damping will be modified for large $t > \epsilon^{-1}$.

Our logic is as follows. We *assume* that the electric field decays with time: $E(x,t) \to 0$, as $t \to \infty$. Then this assumption is shown to be self-consistent by calculating the actual damping rate, $E \propto t^{-1}$, instead of the linear exponential damping.

Assume a periodic boundary condition in x with the period L, and expand the electric field E in a Fourier series. Then, for $k \neq 0$, the second Eq. (1) yields

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$$ikE_k(t) = (2\pi L)^{-1} \int_{-\infty}^{\infty} dv \int_0^L dx\, f(x,v,t) e^{-ikx}$$
$$= (2\pi L)^{-1} \int f_i(a,b) e^{-ikx(a,b,t)} da\, db\,. \quad (2)$$

In Eq. (2), the variables of integration were changed to the Lagrangian variables a and b, the initial position and velocity of a particle. According to the Liouville theorem, the Jacobian of this transformation is unity, and the distribution function is constant along the particle orbit thus reducing to its initial value $f_i(a,b) \equiv f(a,b,0)$.

Equation (2) expresses the electric field in terms of the particle orbit $x(a,b,t)$ defined by $\ddot{x} = E(x,t)$ and the given initial condition, a problem as difficult as the original Eq. (1). However, the integral representation of E in terms of the orbit is very useful for studying the long-time asymptotic, when the electric field is presumably small, and the orbit becomes a motion with a constant velocity, $x(a,b,t) = U(a,b)t$ (plus lower-order terms). The resulting integral of an oscillatory function,

$$E_k(t) \propto \int f_i(a,b) e^{-iktU(a,b)} da\, db, \qquad t \to \infty, \quad (3)$$

for smooth f_i, will generally have only two kinds of asymptotics. If the gradient of $U(a,b)$ is nowhere zero (as, for example, in the linear theory, where $U \simeq b$), then the integral (3) is exponentially small at large t (the Riemann-Lebesgue lemma). If, on the other hand, U has a stationary point where $\partial_a U = \partial_b U = 0$, then the $O(t^{-1/2})$ vicinity of this point dominates the integral, which scales as $E \propto t^{-1}$. Below we show that $U(a,b)$ has stationary points in the general case, and therefore E decays algebraically.

The problem of finding the final velocity U as a function of the initial condition, for a particle moving in a generic decaying potential, is very similar to chaotic scattering [9], and, likewise, due to the transient particle trapping, the function $U(a,b)$ is quite complex (Fig. 1). The fact that $U(a,b)$ is not monotonic is most transparent from the inspection of the particle bouncing at the top (Fig. 2) and at the bottom (Fig. 3) of a decaying potential profile. If the initial potential amplitude is small, the bouncing at the bottom is possible only if ϕ decays sufficiently slowly, e.g., $\phi \propto \epsilon t^{-\alpha}$, $0 < \alpha < 2$, in order that the bounce time $\tau_b \propto \phi^{-1/2}$ be less than t. The initial, linear Landau damping is exponential, seemingly suggesting no bouncing, hence no stationary points of $U(a,b)$ and the persistence of the exponential damping. However, a simple perturbation analysis of the particle motion *near the top* of an evolving potential hill shows that one can always pick initial conditions such that the behavior of Fig. 2 takes place. To some confusion, this turns out possible only if the spatial extrema of $\phi(x,t)$ and $\partial_t \phi(x,t)$ do not coincide, that is, if there is more than just one wave, a safely generic situation. (The result of the left Fig. 1 is for two waves. A similar computation for one exponentially damped wave shows a smooth U with no stationary points.)

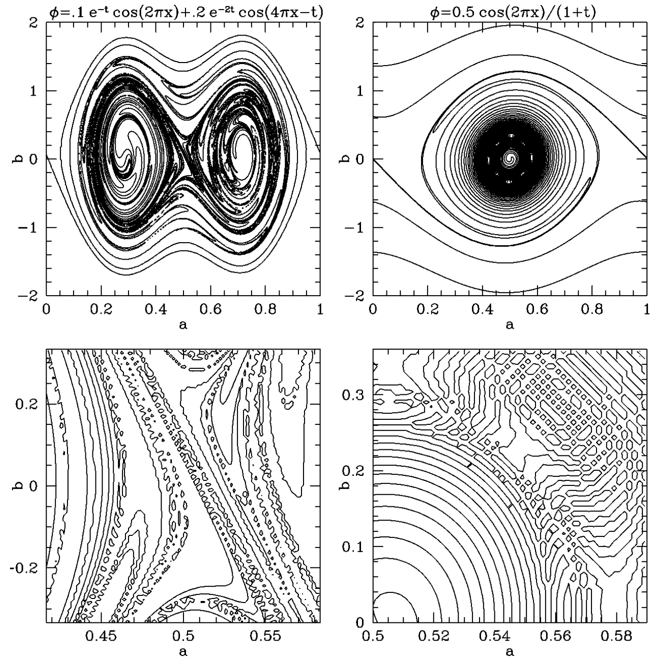

FIG. 1. Contour lines of $U(a,b)$ computed for exponentially (left) and algebraically (right) decaying potentials. The computation was done through the "infinite" time $t = 20$ and $t = 1000$, respectively. The bottom panel is a zoom of the top panel. The presence of multiple extrema (O points) and saddles (X points) of U is apparent.

In fact, $U(a,b)$ has an infinite number of stationary points (a^j, b^j). Upon expanding the particle orbit near such a point at large t, $x(a,b,t) = U^j t + [U^j_{aa}(a - a^j)^2 + U^j_{bb}(b - b^j)^2 + U^j_{ab}(a - a^j)(b - b^j)]t/2 + V^j \ln t + W^j + O(t^{-1})$, Eq. (2) yields the electric field at large t in terms of the infinite series,

$$E_k = \sum_j \frac{f_i(a^j, b^j) e^{-ik(U^j t + V^j \ln t + W^j)}}{k^2 L t [U^j_{aa} U^j_{bb} - (U^j_{ab})^2]^{1/2}} + O\left(\frac{1}{kt^2}\right), \quad (4)$$

which could in principle pose problems in terms of divergencies or cancellations.

The series (4) turns out to be absolutely (exponentially in j) convergent, because it is possible to analyze the accumulation of the stationary points of U. This is due to the adiabaticity of the particle motion at large time, when the bounce frequency $\omega_b \propto \phi^{1/2} \propto t^{-1/2}$ is much larger than the potential damping rate $\dot{\phi}/\phi \propto t^{-1}$. As a result, the adiabatic invariant $J(a,b)$, the (x,v) plane area inside a nearly closed trapped particle orbit, is conserved, and the corresponding angle variable θ is growing with the bounce frequency: $\theta = \int^t \omega_b\, dt \propto t^{1/2}$. Untrapping occurs when the shrinking separatrix of the decaying potential, with the area $S \propto t^{-1/2}$, intersects the orbit with the conserved area J (Fig. 4). For a small $J \propto (a - a_0)^2 + b^2$, the crossing time $t^* \propto J^{-2}$ and the angle

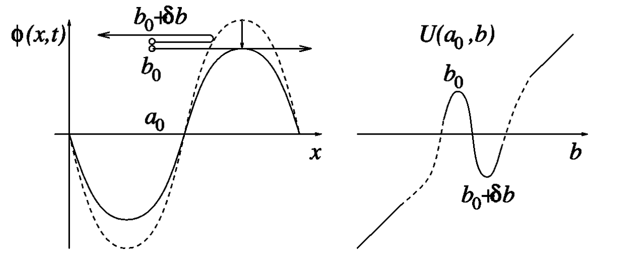

FIG. 2. Particle bouncing in a decaying potential and its signature $U(a,b)$. Near the potential top, an increase in the initial velocity b can bring the particle to the decaying potential barrier earlier, when the barrier was higher, and thus turn the particle around: $U(a_0,b_0) > 0$, $U(a_0,b_0+\delta b) < 0$.

$\theta^* \propto J^{-1}$. Following a small change during the separatrix crossing [10], the adiabatic invariant of the passing particle (now defined as twice the phase-space area) is conserved again and defines the final velocity $|U(a,b)| \simeq J(a,b)/2L$. The sign of U, roughly sgn$(\sin\theta^*)$, depends on whether the crossing happens in the upper or in the lower half-plane of Fig. 4. The width of the steps of U is still finite, $\delta\theta^* \propto e^{-\omega_b t^*} \propto e^{-1/J(a,b)}$; it is determined by the exponentially narrow near-separatrix layer, where the bounce period $2\pi/\omega_b$ diverges logarithmically, and the adiabaticity does not hold. Thus we obtain the approximate analytical expression for the final velocity:

$$U(a,b) \simeq J(a,b)/2L \tanh[e^{1/J(a,b)} \sin J^{-1}(a,b)] . \tag{5}$$

Near the bottom of the well, $a = a_0$, the behavior of Eq. (5) is consistent with the numerical result in Fig. 3. Equation 5 also implies an exponentially growing curvature $U^j_{bb} \propto e^j$ near the steps as one moves to the accumulation point of the U extrema, hence the exponential convergence of the series (4). Thus the long-time behavior of the electric field (4) is dominated by a few "strongest" stationary points of $U(a,b)$. In addition

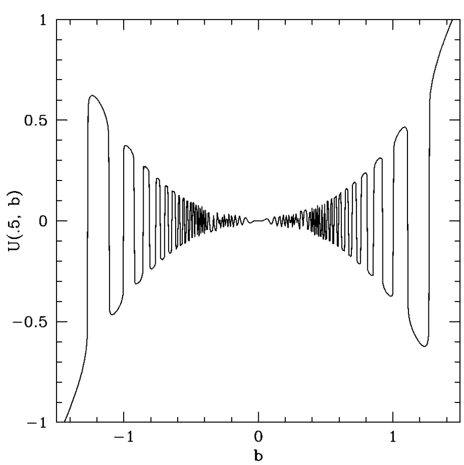

FIG. 3. The cross section of $U(a,b)$ for the algebraically decaying potential shown in Fig. 1. Near the bottom of the potential well, the particle makes many bounces before being released in an essentially random direction.

to the algebraic damping rate, we infer as a by-product the spectrum $E_k \propto k^{-2}$, $k \ll t$, implying the development of steps in the electron density perturbation $\partial_x E$. That is, the shape of the wave changes significantly for $t \gg \epsilon^{-1}$.

In higher dimensions, $d > 1$, asymptotic formula (3) is also valid. If $\mathbf{k} \cdot \mathbf{U}(\mathbf{a},\mathbf{b})$ has stationary points (which we cannot guarantee for $d > 1$), the saddle-point integration predicts a faster damping rate, $E \propto t^{-d}$.

We now turn to the different problem of the relaxation in 2D ideal inviscid incompressible fluid with the velocity $\mathbf{v} = \nabla\psi(x,y,t) \times \hat{\mathbf{z}}$ described by the Euler equation,

$$(\partial_t + \mathbf{v}\cdot\nabla)\omega = 0, \qquad \omega = -\nabla^2\psi . \tag{6}$$

As in the case of Vlasov plasma, we are interested in the long-time relaxation of an initial perturbation $\tilde{\psi}(x,y,t)$ imposed on a stable shear flow $\psi_0(x)$. The deep analogy of this problem with the Landau damping in plasmas has been noted [11,12]. We will assume a periodic boundary condition in y [along the shear flow $v_0(x) = -\psi_0'(x)$]. In linear theory, the perturbation of the stream function ψ is known to be damped, because the reconstruction of ψ from the conserved vorticity ω with growing gradients involves an integration,

$$\psi(x,y,t) = \int G(x,x',y-y')\omega(x',y',t)\,dx'\,dy', \tag{7}$$

where G is the boundary-condition-dependent Green's function with a discontinuous derivative at $(x,y) = (x',y')$. If the flow is unbounded in the x direction, for example, $G_k(x,x') = e^{-|k(x-x')|}$. Unlike plasma waves, the damping law of $\tilde{\psi}$ is algebraic already in linear theory: $\tilde{\psi} \propto t^{-2}$ for monotonic $v_0(x)$ [12–14] and $\propto t^{-1/2}$ for $v_0(x)$ with an extremum [15].

Similar to the Vlasov case, the linear damping in fluid breaks down for large t raising the question of the long-time asymptotic. To this end, we use the same trick as for the Vlasov equation. Upon applying the Fourier transform in y to Eq. (7) and changing the integration variables (x',y') to the Lagrangiam variables (a,b), we obtain

$$\psi_k = \int G_k[x, X(a,b,t)]e^{-ikY(a,b,t)}\omega_i(a,b)\,da\,db\,, \tag{8}$$

where $\omega_i(a,b)$ is the total initial vorticity, and (X,Y) is the orbit of a fluid element with the initial position (a,b).

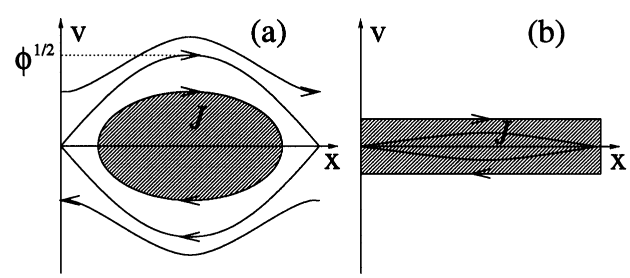

FIG. 4. The adiabatic invariant before (a) and after (b) the separatrix crossing in a decaying potential well.

Consider the case of a smooth and monotonic $v_0(x)$. Then, for a very small perturbation, the unperturbed orbit $(X, Y) \simeq (a, b + v_0(a)t)$ yields an oscillatory integral in a, which is not exponentially small because of the derivative discontinuity in G_k. Changing the variable a to the monotonic $v_0(a)$ and integrating by parts twice then yields $\psi_k(x,t) \propto t^{-2}$ for $k \neq 0$, the well-known linear result. For $v_0(x)$ with a stationary point, the singularity of the Green's function does not matter, and a stationary-phase integration over a yields $\tilde{\psi} \propto t^{-1/2}$, in agreement with [15]. Based on the ordering of terms in Eq. (6) for the regime with $\tilde{\psi} \propto t^{-2}$, Brunet and Warn [15] argued that the nonlinearity remains small and does not change the damping rate. Such an analysis appears superficial, because the accumulation of a small nonlinear effect in the Euler equation is secular. Our analysis of Eq. (8) goes as follows. The flow disturbance of order ϵ makes the orbit essentially depend on both a and b, e.g., $Y = b + v_0(a)t + \epsilon \int v_1(a, b, t, \epsilon)\, dt$. Thus the integral (8) is also oscillatory in b for $t > \epsilon^{-1}$. This is when the nonlinearity comes into effect. Because of periodicity, the phase $ikY(a, b)$ has a stationary point in b producing the additional factor of $t^{-1/2}$ in the integral asymptotic. Finally, for a smooth stable monotonic shear velocity profile, a smooth stream function perturbation decays as $\tilde{\psi} \propto t^{-5/2}$ for $t \gg \epsilon^{-1}$, and, similarly, for $v_0(x)$ with an extremum, $\tilde{\psi} \propto t^{-1}$.

One of the interesting consequences of the nonlinear Landau damping concerns ergodicity, the assumption underlying the Gibbs-ensemble theories of turbulence in the Vlasov-Poisson system [16,17] and in 2D fluid [18–20]. In these theories, the statistical ensemble includes all possible permutations of phase-space (fluid) elements with the associated distribution function f (vorticity ω), via either combinatorial treatment or path integration for the partition function. Such analyses predict specific quantitative results, such as the final relaxed state, for an arbitrary initial condition. In addition to the difficulties with non-Gaussian path integrals [21], the Gibbs-ensemble theory cannot be true in such a generality because of the nonlinear Landau damping. For example, if the initial condition is a slightly perturbed stable shear flow $v_y(x)$, the zonal velocity $v_x(x, y, t)$ will decay as $\epsilon t^{-5/2}$, and the zonal displacement of any fluid element, $\delta X = \int v_x\, dt$, is forever bounded by a small constant: $|\delta X| < C\epsilon$. This purely dynamical fact does not follow from conservation laws alone and implies that the fluid element motion is strongly nonergodic. It follows that the exisiting statistical theories do not work, at least for initial conditions close to stable shear flows. The same is true of the Vlasov-Poisson system, where the velocity change for any particle, $\delta v = \int E(x(t), t)\, dt$, is bounded for infinite time, because $E \propto t^{-1}$, and the convergence of δv is ensured by another power of t coming from the nearly uniform motion in the coordinate $x \simeq Ut$, over which E is zero average. Again, heating up a small group of particles to arbitrarily high energies in an evolving collisionless plasma does not contradict conservation laws; however, this turns out exactly prohibited by self-consistent dynamics.

The shear damping of perturbations is not specific to plane parallel fluid flow; similar results hold for circular monopole vortices developing in the course of long-time turbulent evolution [22,23] and also in the framework of related 2D geophysical fluid equations. In these systems, nonlinear Landau damping is the mechanism of the turbulence relaxation toward large-scale coherent structures. Finally, it appears that decaying 2D turbulence is more about dynamics (vortex merger and the nonlinear damping of vortex perturbations) than statistics.

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