# Exact Two Vortices Solution of Navier-Stokes Equations

O. Agullo and A. D. Verga*
*Institut de Recherche sur les Phénomènes Hors Equilibre,† 12, avenue Général Leclerc, F-13003 Marseille Cedex, France*
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An exact two-dimensional solution of a viscous flow generated by two point vortices is obtained. The viscosity $\nu$ is introduced as a Brownian motion in the Hamiltonian dynamics of two point vortices. The derived exact solution describes in particular the merging process of two Lamb vortices. In the limit $\nu \to 0$, the apparition of a spiral structure in the topology of the vorticity distribution is observed. This solution also describes the selection by viscosity of a particular solution among the infinity of patterns satisfying the Euler equation. [S0031-9007(97)02769-5]

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Vortex interaction is a central issue in two-dimensional fluid turbulence [1]. The basic mechanism, the collision of two localized vortices, was extensively studied mainly using numerical simulations [2], in an attempt to understand the dynamics of coherent structures. Asymptotic analysis also contributed to the study of the motion of vortices in a two-dimensional potential flow, and the results used to design efficient numerical schemes amenable to describe the merging process [3]. A fundamental point about vortex interactions is the appearance of a spiral structure, evolving rapidly and contributing to the generation of small scales. The spiral structure is often invoked to explain the statistical properties of turbulence [4]. In this Letter we show that using a stochastic representation of the Navier-Stokes equations, it is possible to find the exact solution for the interaction of two vortices. The role of the viscosity and the detailed description of the spiral structure are thoroughly investigated.

Let us introduce the basic equations of the model and some useful notations. The vorticity of a two-dimensional flow in terms of the velocity $\mathbf{u}(x_1, x_2, t)$ is given by $\omega = \nabla \times \mathbf{u}$ $[\nabla = (\partial/\partial x_1, \partial/\partial x_2)]$. The incompressibility condition $\nabla \cdot \mathbf{u} = 0$ permits one to introduce the stream function $\psi$ defined by $\mathbf{u} = \nabla\psi \times \hat{\mathbf{e}}_z \equiv \nabla^{\perp}\psi$ and the Poisson equation $\Delta\psi = -\omega$, where the notation $\mathbf{x}^{\perp} = (x_1, x_2)^{\perp} = (x_2, -x_1)$ has been used.

The Navier-Stokes equation written in terms of the vorticity is

$$\frac{D}{Dt}\,\omega \equiv \frac{\partial}{\partial t}\,\omega + (\mathbf{u}\cdot\nabla)\omega = \nu\Delta\omega\,, \tag{1}$$

where $\nu$ is the viscosity.

A classical computation [5], using the Poisson equation, gives the Biot-Savart law for a plane flow evolving in a region $\Omega$

$$\mathbf{u}(\mathbf{x}, t) = \int_{\Omega} \mathbf{K}(\mathbf{x} - \mathbf{y})\omega(\mathbf{y}, t)\, d\mathbf{y}\,, \tag{2a}$$

where $\mathbf{K}(\mathbf{x} - \mathbf{y})$ is the velocity field in $\mathbf{x}$ generated by a vortex of unit circulation located at $\mathbf{y}$. Our main interest is on local interaction processes, independent of the boundary conditions, we take then $\Omega = \mathcal{R}^2$, and the Green function is given by

$$\mathbf{K}(\mathbf{x}) = -\frac{1}{2\pi}\,\frac{\mathbf{x}^{\perp}}{|\mathbf{x}|^2}\,. \tag{2b}$$

A property of the Euler equation [Eq. (1) with $\nu = 0$] is that point vortices are exact solutions, the interaction of a vortex on itself being zero. The vorticity is in that case a sum of Dirac functions, and for two point vortices it reads

$$\omega(\mathbf{x}, t) = \Gamma_1\delta(\mathbf{x} - \mathbf{x}_1(t)) + \Gamma_2\delta(\mathbf{x} - \mathbf{x}_2(t))\,, \tag{3}$$

where $[\mathbf{x}_i(t),\ i = 1, 2]$ represent the positions of the point vortices and $\Gamma_i$ their circulations. Using (2a), the equations of motion for the point vortices are

$$\frac{d}{dt}\,\mathbf{x}_i(t) = \Gamma_j\mathbf{K}(\mathbf{x}_i - \mathbf{x}_j), \qquad i \neq j\,. \tag{4}$$

This system, which can be recast into a Hamiltonian form, is easily integrable, due to the existence of two additional integrals of motion: the relative distance between the two vortices ($\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$) and the position of their center of mass ($\mathbf{M} = \Gamma_1\mathbf{x}_1 + \Gamma_2\mathbf{x}_2$). For a given initial condition, the vortex evolution depends on the value of the total circulation $\Gamma = \Gamma_1 + \Gamma_2$. Either $\Gamma$ is zero and the trajectories of the two point vortices are two parallel straight lines, or $\Gamma$ is nonzero and the trajectories are two concentric circles whose center is the center of mass $\mathbf{M}$.

Our purpose is to generalize these purely nonviscous results to the fully viscous case. The main idea is to construct a stochastic formalism which is a dual representation of the Navier-Stokes equation, in the same way as the Hamiltonian system (4) is a dual representation of the Euler equation. Indeed, when the viscosity term is added to the Euler equation the trajectories of point vortices become stochastic; this is similar to the equation of heat conduction where the Laplacian operator is related to an underlying Wiener process. There is still, in the presence of the viscous term, a solution of the form (3) but the motion of the vortices is given by Langevin equations. In consequence the "characteristics"

  

$[\mathbf{x}_1(t), \mathbf{x}_2(t)]$ associated to the Navier-Stokes equations satisfy Eqs. (4) with added white noise terms [6,7]:

$$\frac{d}{dt}\mathbf{x}_i(t) = \Gamma_j \mathbf{K}(\mathbf{x}_i - \mathbf{x}_j) + \sqrt{2\nu}\,\mathbf{b}_i(t), \qquad i \neq j\,. \tag{5}$$

In Eq. (5) $[\mathbf{b}_i(t), i = 1,2]$ represent two independent white noises (with zero mean and unit variance) defined by $\mathbf{b}_i = d\mathbf{W}/dt$, where $\mathbf{W}$ is the standard Wiener process.

The vorticity (3) also becomes a stochastic process, depending on the paths $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$. Therefore, the problem of finding a solution of the Navier-Stokes equation is equivalent to computing the transition probability $P(\mathbf{x}_1, \mathbf{x}_2, t)$ associated with the stochastic differential equations (5) [7]. To compute $P(\mathbf{x}_1, \mathbf{x}_2, t)$ one must solve a Fokker-Planck equation [8]. The transition probability becomes the basic quantity from which one may calculate, for instance, the mean vorticity distribution $\langle\omega\rangle$.

At this point, some comments are in order. To illustrate the formalism, we apply it to the case of one isolated vortex. For one vortex the probability distribution is the one of a Brownian motion. Indeed, the point vorticity is $\omega(\mathbf{x}, t) = \Gamma\delta(\mathbf{x} - \mathbf{x}_0(t))$, and the motion equation of the vortex is simply $\dot{\mathbf{x}}_0(t) = \mathbf{b}(t)$. The transition probability then satisfies a diffusion equation the solution of which reads

$$P_L(\mathbf{x}_0, t) = \frac{1}{4\pi\nu t}\exp(-|\mathbf{x}_0|^2/4\nu t)\,, \tag{6}$$

assuming that initially the vortex is at the origin. Besides, the vorticity distribution can be obtained using

$$\langle\omega\rangle(\mathbf{x}, t) = \Gamma\int_\Omega P_L(\mathbf{x}_0, t)\delta(\mathbf{x} - \mathbf{x}_0)\,d\mathbf{x}_0 = \Gamma P_L(\mathbf{x}, t)\,, \tag{7}$$

which gives the well-known Lamb vortex [9]. In this case the role of the viscosity is simply related to the diffusion of the initially concentrated vorticity. We will see that for two vortices new basic mechanisms induced by the viscosity appear.

In the light of these results, we can interpret Eq. (5) as describing the interaction of two Lamb vortices. Hereafter, we analyze this system in the simplest case, when both circulations are equal. When $\Gamma_1 = \Gamma_2$, the noise terms in the Langevin equations for $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ and $\mathbf{M} = (\Gamma/2)(\mathbf{x}_1 + \mathbf{x}_2)$ are independent, and hence the two vortex transition probability $P(\mathbf{x}_1, \mathbf{x}_2, t)$ turns out to be the product of the probability distributions of the center of mass $P_\mathbf{M}$, and of the vortex distance $P_\mathbf{r}$: $P(\mathbf{x}_1, \mathbf{x}_2, t) = \Gamma P_\mathbf{M}(\mathbf{M}, t) P_\mathbf{r}(\mathbf{r}, t)$. The knowledge of these probabilities allows the computation of the observed vorticity evolution by direct integration.

Using Eq. (5) we immediately find that the center of mass $\mathbf{M}$ follows a Brownian motion with $P_\mathbf{M}$ a Gaussian probability distribution. Moreover, the distance $\mathbf{r}$ satisfies the Langevin equation

$$\frac{d}{dt}\mathbf{r}(t) = \Gamma\mathbf{K}(\mathbf{r}) + 2\sqrt{\nu}\,\mathbf{b}(t)\,, \tag{8}$$

where $\mathbf{b}(t) = [\mathbf{b}_1(t) - \mathbf{b}_2(t)]/2^{1/2}$ is still a white noise. The Fokker-Planck equation associated to Eq. (8) is [8]

$$\frac{\partial}{\partial t}P_\mathbf{r}(\mathbf{r}, t) = -\Gamma\nabla\cdot[P_\mathbf{r}(\mathbf{r}, t)\mathbf{K}(\mathbf{r})] + 2\nu\Delta P_\mathbf{r}(\mathbf{r}, t)\,. \tag{9}$$

We introduce the polar coordinates $\mathbf{r} = (r, \theta)$, and adimensional units $r_0$, the initial vortex distance, for length, $2\pi r_0^2/\Gamma$ for time, such that the Reynolds number is $\mathrm{Re} = \Gamma/4\pi\nu$. Besides, we suppose $r_0 \neq 0$, otherwise we would just obtain a Lamb vortex of circulation $\Gamma$. In the following $(r, t, \nu)$ will stand for the adimensional quantities where the new $\nu$ is $1/\mathrm{Re}$. Using these notations we rewrite the last equation in the form

$$\frac{\partial}{\partial t}P_\mathbf{r}(r, \theta, t) = -\frac{1}{r^2}\frac{\partial}{\partial\theta}P_\mathbf{r}(r, \theta, t) + \nu\Delta P_\mathbf{r}(r, \theta, t)\,. \tag{10}$$

From expression (10) an important difference between Euler and Navier-Stokes equations can be understood. When $\nu$ is zero, the solution of (8) is in polar coordinates

$$r = 1, \qquad \frac{d}{dt}\theta = 1\,, \tag{11}$$

where $(1, \theta_0)$ is the initial distance, and corresponds to a rotation of period $T = 2\pi$. A probability distribution associated with this deterministic motion is

$$P_\mathbf{r}(r, \theta, t) = \delta(r - 1)\delta(\theta - t - \theta_0)\,. \tag{12}$$

This is of course a solution of (10) with $\nu = 0$. We raise now an important question about the influence of the viscosity. First, we note that the argument of the angular Dirac may be rewritten in the form $\theta - tf(r) - \theta_0$, where $f$ is an arbitrary function satisfying $f(1) = 1$. Second, for any arbitrary small viscosity (in fact, $\nu \to 0^+$) the effect of diffusion will spread out the $\delta$ functions in (12). Therefore, the actual distance (in the probabilistic sense) between the vortices will be slightly different to $r_0 = 1$, which raises the problem of the choice of $f$. This is precisely the role of the viscosity, which will select a particular form of $f$, compatible with the Navier-Stokes equation, as we show below.

We now compute the solution of (10) with initial condition, given in polar coordinates

$$P_\mathbf{r}(r, \theta, 0) = \delta(r - 1)\delta(\theta - \theta_0)\,, \tag{13}$$

which means that the initial distance between the two vortices is $(1, \theta_0)$. The Fokker-Planck equation being invariant by rotation, one can choose $\theta_0 = 0$. Moreover, this equation is separable in radial and angular variables.

Using the linearity of the equation, we look for a solution in the form of a Fourier series

$$P_{\mathbf{r}}(r,\theta,t) = \sum_{p\in Z} a_{p,\lambda} e^{-\lambda^2 t} e^{ip\theta} g_{p,\lambda}(r),$$

where $\lambda^2 > 0$ to ensure convergence and $a_{p,\lambda}$ are coefficient to be determined by the initial condition (13). The radial equation for $g_{p,\lambda}(r)$ reduces to a Bessel differential equation. Using a representation of the initial condition in a base of Bessel functions and comparing to the solution of the radial equation, we obtain the appropriated form of the coefficients $a_{p,\lambda}$. The general solution is finally written as

$$P(r,\theta,t) = \sum_{p\in Z} e^{ip\theta} \int_0^{+\infty} \lambda\, d\lambda\, e^{-\lambda^2 t} J_{\mu_p}(\lambda r) J_{\mu_p}(\lambda),$$

where $\mu_p^2 = ip/\nu + p^2$, or after integration [10], we obtain the explicit formula

$$P_{\mathbf{r}}(r,\theta,t) = G(r,t) \times \left[1 + 2\Re \sum_{p=1}^{+\infty} I_0^{-1}\left(\frac{r}{2\nu t}\right) I_{\mu_p}\left(\frac{r}{2\nu t}\right) e^{ip\theta}\right], \qquad (14)$$

where $\Re$ stands for the real part, $I_{\mu_p}$ are modified Bessel functions, and

$$G(r,t) = \frac{1}{4\pi\nu t} e^{-\frac{r^2+1}{4\nu t}} I_0\left(\frac{r}{2\nu t}\right). \qquad (15)$$

The vorticity field associated to (14) and the Gaussian $P_{\mathbf{M}}$ is an exact solution of the two-dimensional Navier-Stokes equation, this is the main result of this Letter. A straightforward computation allows us to verify that (14) is effectively a normalized positive probability distribution.

The probability $G(r,t)$ is the axisymmetric part of $P_{\mathbf{r}}$, and can be also obtained as the solution of the heat equation with initial condition a distribution of the vorticity along a ring, it characterizes the diffusion of the radial distance. It is worth noticing that this axisymmetric part becomes asymptotically dominant in the limit $t \to +\infty$ [$I_\mu(0) = \delta_{0,\mu}$, the Kronecker $\delta$], demonstrating that the final state of the system is isotropic. Therefore, the solution (14) describes the change in the topology of the flow, the initial localized two vortices evolve to an axisymmetric structure.

An important property of the Bessel functions $I_\mu(x)$ is that they are all equivalent in the limit $x \to +\infty$, independently of the order $\mu$. Formally, if we take as the common limit $I_0$, one can write

$$P_{\mathbf{r}}(r,\theta,t) \sim G(r,t)\delta(\theta), \qquad (t \to 0), \qquad (16)$$

which gives the expected limit (13) as $t \to 0$. In conclusion, (14) is the Green function of the Fokker-Planck equation (10) and gives the temporal evolution of the distance between the two initial point vortices. Of course, the limiting case $\mathrm{Re} = 1/\nu = 0$ gives $\mu_p = p$ and the solution becomes $P_L(\mathbf{r} - \mathbf{r}_0, t)$. The evolution of the distance is therefore a Brownian motion, as one could also derive using the Langevin equation.

The general formula is quite abstract and, because of the Bessel functions of complex order, its numerical evaluation is not easy. However, some useful information can be obtained in the limit of small viscosity. Using the asymptotic expansion of the Bessel functions, an approximation of (14) is given by

$$P_{\mathbf{r}}(r,\theta,t) \sim G(r,t)\Theta(r,\theta,t), \qquad \nu \to 0, \qquad (17)$$

where the asymmetric part of (17) is

$$\Theta(r,\theta,t) = \sqrt{\frac{\pi r}{2\nu t}} \sum_{p=-\infty}^{+\infty} e^{-\frac{\pi^2 r}{\nu t}[p - \frac{1}{2\pi}(\theta - \frac{t}{r})]^2}, \qquad (18)$$

the Theta elliptic function [10]. The function $\Theta$ is a solution of the one-dimensional heat equation in the circle. In the present context, it describes the development of a spiral structure, triggered by the interaction of the vortices (whose cores are of finite extent for $t > 0$). We note that the spiral stretching increases as $t$ while the diffusion goes as $t^{1/2}$, then the development of spiral structure is faster than diffusion. This may be easily seen in the limit $\nu \to 0$

$$\Theta(r,\theta,t) = \delta(\theta - t/r), \qquad (19)$$

which corresponds to a concentration of the probability on a spiral (in the original units) $S : \theta \to \theta(r) = \Gamma t/2\pi r_0 r$ centered at $r = 0$ and spreading in time. The emergence of this spiral structure (decreasing as $1/r$) is the one selected by the viscosity [selection of $f(x) = 1/x$] coupled with the rotation effect ($\Gamma \neq 0$). Therefore, the viscosity is not only important for its effect on diffusion and topology change, but also in selecting a particular spiral structure. The particular form of $f$, at least for small viscosity, may be related to the conservation of the angular velocity $v_\theta = r\dot{\theta} = \Gamma/2\pi r_0$. Expression (18) adds a "Gaussian thickness" to the spiral structure. Of course, if we make $\nu = 0$ for the radial and nonradial parts of $P_{\mathbf{r}}$, we get the expected limit, the evolution of the distance between two point vortices (12).

Besides, the probability distribution on the spiral structure follows the law $G(r,t)$ and has a sharp maximum at $r_p(t)$, solution of the implicit equation (returning to adimensional quantities),

$$r_p = \frac{I_1(\frac{r_p}{2\nu t})}{I_0(\frac{r_p}{2\nu t})} \leq 1. \qquad (20)$$

For $t = 0$, $r_p = 1$ and decreases to zero as time grows. The thickness of this maximum is given by $(\nu t)^{1/2}$, and is narrow in the limit examined. According to the position of this peak on the spiral, different behaviors may be obtained, as can be seen with a detailed study of the spiral structure.

In Fig. 1 the spiral $S$ is plotted at three different times. We choose the Reynolds number $\mathrm{Re} = 1/\nu = 100$ in

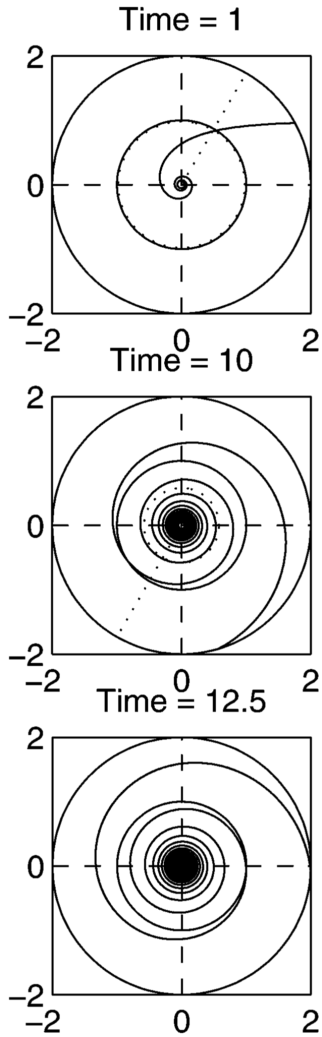


FIG. 1. Plot of the spiral $S$ at $t = 1$, 10, and 12.5. Dotted circle: $r = r_p = 0.98$, 0.59, and 0.09; solid circle: at $r = r_0 = 1$; dotted line: $\theta = \theta_p$.

such a way as to observe the merging of two vortices during a valid approximation time. At the beginning $t = 1$, the distance distribution is closely concentrated around $r_p(1) \sim 0.98 \sim 1$: The two vortices have mutually rotated by $\theta_p \sim \pi/3$ but their distance is practically the initial one. The vortices are still clearly separated. The coupling effects between rotation and diffusion are still negligible. Therefore, at short times the vorticity distribution corresponds to two well-separated Lamb vortices subject to rotation ($\Gamma \neq 0$).

At time $t = 10$, $r_p \sim 0.59$ and $\theta_p \sim 4\pi/3$. The distance distribution is essentially concentrated on the spiral with radial values $r \in [r_p - (\nu t)^{1/2}, r_p + (\nu t)^{1/2}]$, that is to say angular values $\theta \in [2\pi + 1.6\pi, 10\pi + 1.5\pi]$: This corresponds to the development and stretching of vortex arms, which wind around the other vortex. Here, each arm winds the other vortex on several rotations.

After this transitory period, $r_p$ quickly decreases and at $t = 12.5$, $r_p \sim 0.09$: The spiral is tightly bounded around $r_p$ and becomes similar to circles. At this time the angular isotropy is fully developed, $\Theta$ is almost uniform in the region where the probability is concentrated showing the disappearance of the spiral arms, and the merging is ending. Lastly, $r_p \sim 0$ and the further evolution is purely diffusive.

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*Electronic address: verga@marius.univ-mrs.fr
†Unité Mixte de Recherche 138 of the Centre National de la Recherche Scientifique, Universités d'Aix-Marseille I et II.

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