

Active Stabilization, Quantum Computation, and Quantum State Synthesis

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Active stabilization of a quantum system is the active suppression of noise (such as decoherence) in the system, without disrupting its unitary evolution. Quantum error correction suggests the possibility of achieving this, but only if the recovery network can suppress more noise than it introduces. A general method of constructing such networks is proposed, which gives a substantial improvement over previous fault tolerant designs. The construction permits quantum error correction to be understood as essentially quantum state synthesis. An approximate analysis implies that algorithms involving very many computational steps on a quantum computer can thus be made possible. [S0031-9007(97)02672-0]

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All physical systems are subject to noise, arising from an unavoidable coupling to an environment which cannot be completely analyzed or controlled. Typically, noise is a problem we wish to minimize, and for this purpose passive and active stabilization can be applied. By *passive* stabilization we mean the careful construction and isolation of a system so as to reduce the noise level to a minimum. By *active* stabilization we mean the use of detection and feedback to suppress the tendency of a system to depart from some desired state. An early example is the governor in a steam engine, and a common modern example is the electronic servo based on a stable voltage reference, a high-performance amplifier and a low-noise resistor (see inset to Fig. 2). Passive stabilization is insufficient to stabilize any but the most simple devices, whereas active stabilization is a very powerful method and is used throughout the natural world, whether in man-made devices or in living organisms.

The above examples of active stabilization are classical, not quantum systems, however. Is it possible to apply active stabilization in quantum physics? That is, can the complete unitary evolution of the state throughout its Hilbert space be actively stabilized against the effects of random noise? It has been widely supposed that the answer to this question is “no.” The reason is because active stabilization is based on amplification and dissipation. However, a general unknown quantum state cannot be amplified (“no cloning theorem” [1]), and dissipation will prevent unitary evolution, so it would seem impossible to stabilize a quantum state.

This supposed impossibility was recently called into question by the concept of quantum error correction (QEC) [2–9], which has shown that very powerful techniques exist for restoring a quantum state which has been affected by noise, and error correction can be applied in a manner (dubbed “fault tolerant” [10,11]) which is itself sufficiently insensitive to noise during the corrective process to allow an overall stabilization of a quantum system.

In this Letter I will present a reinterpretation of QEC, showing that the task of applying QEC in the lab is

essentially one of *quantum state synthesis*, that is, the preparation of a desired (nontrivial) quantum state. This is more than merely a change of emphasis, since the proposed method allows a substantial improvement on the best previously described correction technique. I will then estimate the degree of passive stability and redundancy needed to enable a quantum computer to carry out long computations.

First let us summarize the concept of QEC. Suppose we wish to store or communicate an unknown quantum state $|\phi\rangle$ of some quantum system q , in the presence of noise. We introduce an extra system c , similar to q , in an initial known state $|0\rangle$. A unitary “encoding” operation E is performed, $|\phi\rangle \otimes |0\rangle \rightarrow E(|\phi\rangle \otimes |0\rangle) \equiv |\phi_E\rangle$. The encoded state $|\phi_E\rangle$ is transmitted or stored, during which time it is subject to noise, $|\phi_E\rangle \rightarrow e_s |\phi_E\rangle$. Now, such noise will not degrade the transmitted quantum information, for all error operators e_s such that [12]

$$E^\dagger e_s |\phi_E\rangle = |\phi\rangle \otimes |s\rangle. \quad (1)$$

Furthermore, the “decoding” operation E^\dagger will recover the correct quantum state $|\phi\rangle$ after noise processes of the form $|\phi_E\rangle \rightarrow \sum_s e_s |\phi_E\rangle$, and even after mixtures of such processes [4,5,8,9,13]. The latter possibility includes the case of entanglement with an unknown environment, which is an important source of error. The theory of QEC involves identifying encodings E such that the set $S = \{e_s\}$ of correctable errors includes those *most likely* to be produced by the actual noise to which the system is subject.

In the rest of this Letter, it will be convenient to consider the case that q and c are sets of two-state systems (quantum bits or qubits). Let k be the number of qubits in q , and $n - k$ be the number of qubits in c . The combined system will be referred to as qc (also a mnemonic for “quantum channel”). A general error of a single qubit can be expressed as a sum [13] of operators taken from the set $\mathcal{P} = \{I, \sigma_x, \sigma_z, \sigma_x \sigma_z = i\sigma_y\}$, where I is the identity (corresponding to no error) and σ_i are the Pauli spin

operators. A general error of n qubits is a sum of tensor products of such single-qubit errors.

Each type of noise will have a corresponding quantum error correcting code (QECC) designed to deal with it. In what follows we will analyse the case where the noise involves all the Pauli spin operators, but affects different qubits *independently*. This is sufficient to cover many realistic situations. Let ϵ be the order of magnitude of erroneous terms in the density matrix of qc caused by the noise. A t -error correcting code is defined to be one which can correct any error where up to but no more than t qubits out of the n are defective. This allows the decoding operation to be successful [Eq. (1)] for all the terms in the noisy density matrix involving up to t defective qubits. The remaining uncorrected part has a relative magnitude of order $\epsilon^{(t+1)}C(n, t + 1)$, which is very small for $\epsilon n < t$ and $t \gg 1$, so the noise is strongly suppressed [$C(n, t)$ is the binomial coefficient $n!/t!(n - t)!$]. Such powerful QECC's were first discovered by Calderbank and Shor [4] and myself [5]; they are related to the pioneering work of Shor [2] and myself [3].

So far we have discussed QEC as it might be applied to a communication channel, in which the noise occurs in the channel but is not present in the operations E and E^\dagger . To consider active stabilization, it is necessary to relax this assumption. We will require the *recovery operator* R , defined by $R e_s |\phi_E\rangle \otimes |0\rangle_a = |\phi_E\rangle \otimes |s\rangle_a, \forall e_s \in S$. This is an interaction between qc and an ancilla a consisting of further n_a qubits introduced in an initial state $|0\rangle_a$. The recovery corrects the encoded system qc without decoding it, carrying the noise into a .

An example quantum network to perform recovery is shown in Fig. 1. This network is for a $[[n, k, 2t + 1]] = [[5, 1, 3]]$ single-error correcting code in which a single qubit is encoded into five [6,7]. The network can be obtained directly from the *stabilizer* of the code [14,15].

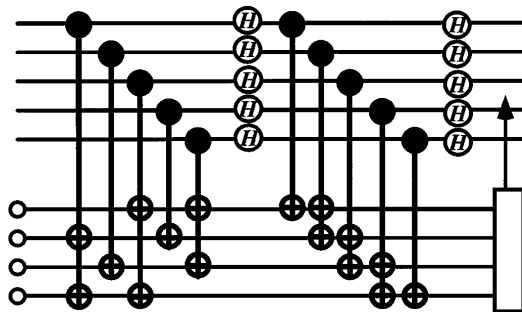


FIG. 1. Recovery network for a $[[5, 1, 3]]$ code, showing how the network may be built directly from the stabilizer matrix, but yielding a design which functions poorly in the presence of noise. The symbol \textcircled{H} means the Hadamard transform $|0\rangle \rightarrow (|0\rangle + |1\rangle)/\sqrt{2}, |1\rangle \rightarrow (|0\rangle - |1\rangle)/\sqrt{2}$. The rectangular box represents a measurement of the ancilla bits; the vertical arrow signifies a final corrective operation which depends on this measurement result.

In this case the stabilizer is, in the notation of [14],

$$\mathcal{H} = (\mathcal{H}_x | \mathcal{H}_z) = \left(\begin{array}{ccc|ccc} 11000 & 00101 \\ 01100 & 10010 \\ 00110 & 01001 \\ 00011 & 10100 \end{array} \right). \quad (2)$$

The quantum exclusive-or (XOR) gates shown in Fig. 1 are positioned exactly where the 1's occur in the two halves of the stabilizer. Their effect is the transformation $e_s |\phi_E\rangle \otimes |0\rangle_a \rightarrow e_s |\phi_E\rangle \otimes |s\rangle_a$. The final state of a is measured, to reveal the error syndrome s , which is used to identify the corrective operation e_s^{-1} to be applied to qc (see [5,6,11] for details).

The network in Fig. 1 will not function well in the presence of noise, since errors in one qubit can be transported to several others by the XOR operations, and errors in a go uncorrected. I now propose to replace this general type of recovery network by the type illustrated in Fig. 2. The new network requires a larger ancilla, $n_a = 2n$ instead of $n - k$, but has the great merit of reducing interactions between a and qc to a minimum, and furthermore each qubit in a only interacts with a single qubit in qc , so single qubit errors in a are only carried onto single qubits in qc . The latter feature was also achieved in the fault-tolerant design of DiVincenzo and Shor [11], but their technique involved on the order of $(2t + 1)(n + k)$ interactions between a and qc , whereas the present method involves just $2n$ interactions; this is important because each interaction introduces noise into qc .

The network of Fig. 2 is constructed as follows. First, the ancilla is prepared in a superposition of all states satisfying the parity checks of \mathcal{H} , where each row of \mathcal{H} is interpreted as a single $2n$ -bit parity check. Next, a and qc interact in such a way that errors in qc are carried into a . The idea is to store the error syndrome not directly in the state of $n - k$ qubits of a as before, but in the value of the $n - k$ parity checks on a given by \mathcal{H} . It is simple to show that any error e_s (acting on qc before the recovery

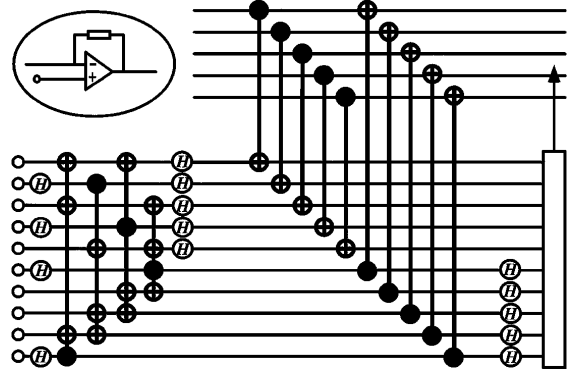


FIG. 2. Proposed design for the recovery network, illustrated for the same code as Fig. 1 [cf. Eq. (2)]. The interaction between the encoded system and the ancilla is now minimal, and prevents unnecessary propagation of errors. Inset: the elements of a classical active stabilization servo.

network is applied) will result, after the action of the network, in a final state of a which passes or fails these parity checks in the correct way to yield the error syndrome. Furthermore, the $2n$ qubits of a were prepared in an equally weighted superposition of $2^{2n-(n-k)}$ orthogonal product states, in a Hilbert space of 2^{2n} dimensions, so there is sufficient Hilbert space remaining for a to store $n - k$ qubits of quantum information, which is just enough to store the error syndrome. Therefore, no further information passes from the qc to a , and there is no problem of extraneous entanglement left between the two systems after their interaction.

We have now reduced the problem of QEC essentially to one of quantum state synthesis, since the only nontrivial part of the recovery network is that involved in preparing the ancilla state. Comparing with active stabilization of classical systems (Fig. 2), we see that the amplifier is replaced by the use of redundant storage of the quantum information, the stable reference is replaced by a precise quantum state synthesis, and the dissipative feedback resistor is replaced by dissipative measurements on the ancilla (or, if preferred, by a unitary network to interpret the syndrome, followed by dissipative re-preparation of a).

The preparation of a can be accomplished in the presence of noise by adopting the ideas of purification [16] and fault-tolerant quantum computing [10]. The ancilla is prepared and then tested before it is allowed to interact with qc . If it fails a test, the preparation is started over again. Only once a "good" ancilla state is obtained is the rest of R applied. The tests could consist of measurements of the parity checks which a should satisfy, whether in the x or z bases, performed by XOR's onto an additional qubit introduced for the purpose. Of course noise will cause some of these tests to give erroneous results, but random errors in a are unlikely to go undetected, except those occurring during or after the last test operation on each qubit in a . Since there are n_a qubits, there are n_a such opportunities for error. These errors will be taken into account in the analysis to follow, the most questionable assumption being that they are distributed independently amongst the qubits of a .

Noise will also take place during the interaction of qc and a , and at other times, resulting in an inappropriate syndrome in a . To handle this, the whole syndrome generation procedure (preparation, interaction, measurement) is repeated r times (following Shor [10]). Together the r syndromes are used to deduce the corrective operation most likely to be appropriate for qc after the r th syndrome was generated. It will be assumed that the probability this final corrective operation is nevertheless wrong is equal to the probability that more than half of the r syndromes were wrong. Note that the $O(r)$ correct syndromes will not necessarily all be the same; each one identifies the errors in qc at the time it was generated.

The method of Fig. 2 becomes particularly elegant when the encoding uses a $[[n, k = 2k_c - n, d]]$ QECC based on a $[n, k_c, d]$ classical weakly self-dual code (i.e., the codes

first discussed in [4,5]). This is shown in Fig. 3. For these codes, the correction of σ_x and σ_z errors can take place separately, using an ancilla of n qubits twice. Remarkably, the preparation of a is identical to first preparing it in the encoded zero state of qc , $|0_E\rangle$, and then carrying out a Hadamard transform! Figure 3 shows two separately encoded qubits interacting by a quantum gate, which we take to be one elementary step in a longer quantum computation, followed by a recovery operation applied first to one qubit, then to the other. Not shown is the testing of a , nor the repetition of the syndrome generation.

To estimate the effects of noise in the whole network shown in Fig. 3, a simple analysis using classical error probabilities will suffice. This relies on the more thorough treatment of Knill and Laflamme [8,17] who have shown that such an analysis is sufficient for approximate purposes as long as the noise does not have certain pathological features.

Let α be the probability that the syndrome obtained in any one cycle of syndrome generation does not correctly indicate the errors in qc . During each cycle there are $n_a = n$ opportunities for the last tests on a to leave errors in a , n XOR gates between qc and a , and n measurements on a , making $3n$ opportunities in all for errors in a , so $1 - \alpha = [(1 - \gamma)(1 - \epsilon)^n]^{3n}$, giving $\alpha = 3n(\gamma + n\epsilon)$ for $\gamma, \epsilon, \alpha \ll 1$, where γ is the probability of gate or measurement failure and ϵ is the probability per time step that a freely evolving qubit defects. A failed gate causes all qubits involved in the gate to become defective; a failed measurement yields an arbitrary result. The ϵ term allows one time step per gate; that is, these gates are *not* performed in parallel. The probability that more than half the r syndromes are wrong is

$$P_1 \cong \sum_{i=(r+1)/2}^r C(r, i) \alpha^i. \quad (3)$$

How many errors will accumulate in qc ? The n XOR's between qc and a during each cycle carry all the errors already in a into qc , and add further n opportunities for error, making $r2n$ in all. The original elementary step in the computation involves $O(n)$ logic gates, using fault-tolerant computation [10]. Finally we must add further $r2n$ error opportunities which each half (either σ_x or σ_z

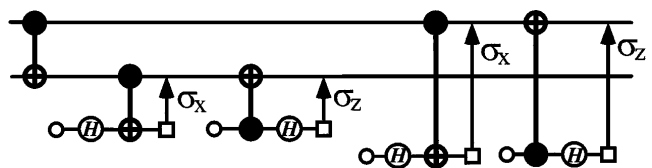


FIG. 3. A complete computational step, including stabilization, for two encoded qubits. In this diagram each horizontal line is a single encoded qubit, i.e., n physical qubits, and an open circle at the beginning of a line means the preparation of the encoded zero state $|0_E\rangle$. The vertical arrow represents the corrective operation σ_x or σ_z carried out on one or more qubits.

correction) of the recovery network causes for the other half. Given that qc is encoded with a t -error correcting code, the probability that more errors accumulate in qc than can be corrected is

$$P_2 \cong \sum_{i=t+1}^{n[4r+O(1)]} C(n[4r+O(1)], i)(\gamma + n\epsilon)^i. \quad (4)$$

The probability that the whole computational step fails is $p = 4(P_1 + P_2)$, assuming the two encoded qubits can be recovered in parallel. Solving this equation for p as a function of γ , for given n, t , taking $\epsilon = \gamma/10n$ (i.e., the noise is dominated by gate and measurement errors), and choosing r large enough to make $P_1 < P_2$, yields $p = O((nr\gamma)^{(t+1)})$. The solution is plotted in Fig. 4 for various QECC's taken from [18]. The main result is that γ must be reduced to a level less than $\sim 10^{-4}$ before active stabilization can work, but below this break-even point the stabilization is very powerful, allowing, for example, $p = 10^{-12}$ with γ of order 10^{-5} . For a computation involving S elementary steps, it is only necessary to reduce p to less than $1/S$ for the stabilization to be deemed sufficient, since then a repetition of the whole computation can be used to enhance the chances of getting a correct result. Thus $\gamma \approx 10^{-5}$ would permit a quantum computation involving 10^{12} steps, which is probably impossible to achieve with passive stabilization alone.

In conclusion, active stabilization of a quantum system is possible through the use of quantum error correction, as long as the recovery network removes more noise than it introduces. This has been achieved by minimizing the interaction between the ancilla and the system to be stabilized, using only $2n$ quantum XOR gates, which also

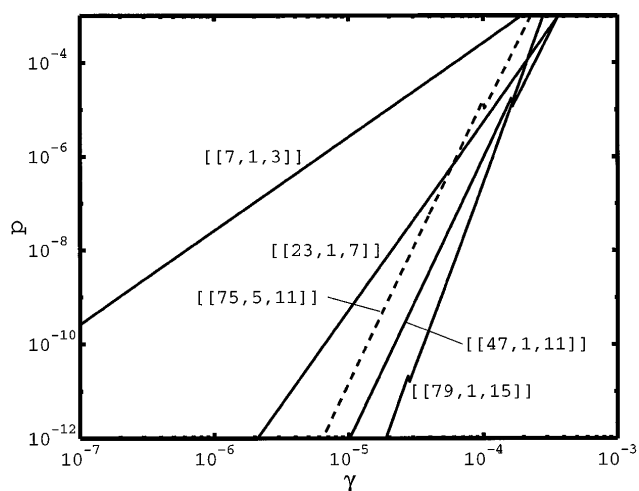


FIG. 4. Solution of $p = 4(P_1 + P_2)$; $\epsilon = \gamma/10n$ [cf. Eqs. (3) and (4)], as a function of γ , for various quantum codes which support fault tolerant computation. The kinks in the curves occur when r must be increased in order to keep $P_1 < p$; the values of r are in the range 3 to 15. The codes are identified by the notation $[[n, k, 2t + 1]]$. The dashed line is for a higher rate code which probably will support universal fault tolerant computation, but this has yet to be proved.

prevents errors propagating from one qubit to many. The recovery network consists primarily in the preparation of the ancilla state, and the syndrome information is stored in a subtle form, in the values of parity checks over the ancilla qubits. A heuristic analysis of the effects of noise on the whole recovery network suggests that long quantum computations can thus be made possible. The method described is a great improvement on the previously reported fault-tolerant error correctors, and allows QEC to be reinterpreted as active stabilization based on repeated quantum state synthesis. Future work must analyse in more detail this state synthesis (ancilla preparation) in the presence of noise.

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