Self-Consistent Wormhole Solutions of Semiclassical Gravity

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We present the first results of a self-consistent solution of the semiclassical Einstein field equations corresponding to a Lorentzian wormhole coupled to a quantum scalar field. The specific solution presented here represents a wormhole connecting two asymptotically spatially flat regions. In general, the diameter of the wormhole throat, in units of the Planck length, can be arbitrarily large, depending on the values of the scalar coupling ξ and the boundary values for the shape and redshift functions. In all cases we have considered, there is a fine structure in the form of Planck-scale oscillations or ripples superimposed on the solutions. [S0031-9007(97)02570-2]

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Wormholes are topological handles in spacetime linking widely separated regions of a single universe or "bridges" joining two different spacetimes. Interest in these configurations dates back at least as far as 1916 [1] with punctuated revivals of activity following both the classic work of Einstein and Rosen in 1935 [2] and the later series of works initiated by Wheeler in 1955 [3]. More recently, a fresh interest in the topic has been rekindled by the work of Morris and Thorne [4], leading to a flurry of activity branching off into diverse directions. A brief resumé of current work devoted to the physics of Minkowski-signature wormholes includes topics addressing fundamental features of traversable wormholes [4,5], explicit modeling of wormhole metrics and the corresponding classical [6] and quantum mechanical stability [7] analyses, wormholes as time machines and the problem of causality violation [8], wormholes in higher-derivative gravity [9], wormholes from the gravitationally squeezed vacuum [10], possible cosmological consequences of early universe wormholes [11,12], and wormholes as gravitational lenses [13]. A thorough and up-to-date survey of the present status of Lorentzian wormholes may be found in the excellent monograph by Visser [14].

There are plausible physical arguments suggesting that Lorentzian wormholes should exist at least at scales of order the Planck length. Most of what is known about them is based on detailed analyses of models, and within the literature devoted to the subject, the existence of wormholes is taken as a working hypothesis. Metrics describing wormholes with desirable traits are written down by fiat, and the properties of the corresponding hypothetical stress-energy tensors needed to support the wormhole spacetime are then worked out and analyzed. In an example of an analysis of this sort, Ford and Roman [15] have derived approximate constraints on the magnitude and duration of the negative energy densities which must be observed by a timelike geodesic observer in static spherically symmetric wormhole spacetimes. More recently, Taylor, Hiscock, and Anderson have argued that stress tensors for massive minimally and/or conformally coupled scalars fail to meet the requirements for maintaining five particular types of static spherically symmetric wormholes, but have not solved the back-reaction problem [16]. In particular, no one up to now has succeeded in writing down a bona fide wormhole *solution* of either the classical or semiclassical Einstein field equations. The reason for this state of affairs is easy to understand. In the first case, it is well known that any stress energy that might give rise to a wormhole must violate one or more of the cherished energy conditions of classical general relativity [4,5]. Hence wormholes cannot arise as solutions of classical relativity and matter. If they exist, they must belong to the realm of semiclassical or perhaps quantum gravity. In the realm of semiclassical gravity, one sets the Einstein tensor equal to the expectation value of the stress-energy tensor operator of the quantized fields present,

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle\,. \tag{1}$$

A primary technical difficulty in semiclassical gravity is that $\langle T_{\mu\nu}\rangle$ depends strongly on the metric and is generally difficult to calculate. Until recently, all calculations of $\langle T_{\mu\nu}\rangle$ have been performed on fixed classical backgrounds. The fixed background in turn, as its name implies, must be a solution of the classical Einstein equation. As there are no classical wormhole backgrounds, no corresponding semiclassical back-reaction problem can be set up meaningfully.

In this Letter we present and summarize the results of the first *self-consistent* wormhole solutions of semiclassical Einstein gravity. Prior to this, a self-consistent wormhole solution had been obtained using a phenomenological stress tensor not derived from quantum field theory [17]. The results of the present calculation may be taken as numerical evidence for the existence of Lorentzian wormholes. For the source term in (1) we employ the stress-energy tensor of Anderson, Hiscock, and Samuel, which is calculated for a quantized scalar field in an arbitrary static and spherically symmetric spacetime [18]. This means that in the field equation (1), both the Einstein tensor as

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well as the stress tensor individually depend on two independent functions of the radial coordinate. When supplemented with the appropriate set of boundary conditions, the solutions of the resultant coupled nonlinear differential equations are therefore self-consistent because both the spacetime metric and the distribution of stress energy are determined simultaneously and coherently. This should be contrasted clearly with, and distinguished from, the approach taken in perturbative back-reaction problems in which the (background) spacetime is fixed once and for all, and the stress tensor is supplied as an explicitly known function of that fixed background.

In the following, we set up the semiclassical field equations valid for any static and spherically symmetric spacetime containing quantized scalar matter and discuss the nature of the boundary conditions needed for solving this system of fourth-order equations. We consider the case of a conformally coupled scalar field. Results of the numerical calculations are presented graphically, and we appeal to an approximate but analytic treatment to reveal the important asymptotic behavior of the solution. Further details of this and related calculations will appear in a separate paper.

The metric for a general static and spherically symmetric spacetime can be cast into the form

$$ds^2 = -f(l)dt^2 + dl^2 + r^2(l)\,(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (2)$$

where $f(l), r(l)$ are two independent functions of the proper distance l. This form of the metric is suitable for dealing with wormholes, or for that matter, any static and spherically symmetric spacetime which might contain a throat, i.e., $r(0) > 0$. Following the established nomenclature, f is denoted the redshift function and r is known as the shape function [14]. In this metric, the semiclassical field equations take the form [19]

$$G_\mu^\nu[f(l), r(l)] = 8\pi \langle T_\mu^\nu[f(l), r(l); \xi]\rangle, \quad (3)$$

where ξ is the (nonminimal) scalar coupling to the metric. Note the dependence of both sides of Eq. (3) on the two unknown functions. In the metric (2), the components of the Einstein tensor are given by $G_t^t = 2r''/r + r'^2/r^2 - 1/r^2$, $G_l^l = f'r'/fr + r'^2/r^2 - 1/r^2$, and $G_\theta^\theta = f''/2f + r''/r + f'r'/2fr - f'^2/4f^2$; the prime denotes the derivative with respect to l. An accurate analytic approximation to the exact numerically calculated scalar field stress energy tensor was developed in Ref. [18] and is expressed there as

$$\langle T_\mu^\nu\rangle_{\rm analytic} = (T_\mu^\nu)_0 + (\xi - \tfrac{1}{6})(T_\mu^\nu)_1 + (\xi - \tfrac{1}{6})^2 (T_\mu^\nu)_2 + (T_\mu^\nu)_{\log}, \quad (4)$$

where the individual factors $(T_\mu^\nu)_{0,1,2}$ are written in terms of two functions of the radial coordinate. The last factor $(T_\mu^\nu)_{\log}$ was left in terms of combinations of curvature tensors and covariant derivatives [20]. To be of practical use in the present calculation, we must work out these curvature terms and transform all the factors in (4) in terms of our two functions $f(l), r(l)$ of the proper distance l. Carrying out these straightforward but rather lengthy steps we find that the semiclassical Einstein equations for a conformally coupled scalar ($\xi = \frac{1}{6}$) are as follows ($K^2 = \frac{1}{5760\pi}$) (ll component only):

$$\begin{aligned}\frac{f'r'}{fr} + \frac{r'^2}{r^2} - \frac{1}{r^2} = K^2\Bigg[& \frac{f'^4}{f^4} - 16\frac{f'^3r'}{f^3r} + 64\frac{f'r'^3}{fr^3} - 4\frac{f'^2r''}{f^3} + 64\frac{f'r'f''}{f^2r} - 64\frac{r'^2f''}{fr^2} - 4\frac{f''^2}{f^2} - 48\frac{f'^2r''}{f^2r} + 32\frac{f'r'r''}{fr^2} \\ & + 32\frac{f''r''}{fr} + 8\frac{f'f'''}{f^2} - 32\frac{r'f'''}{fr} - 32\frac{f'r'''}{fr} + \ln f\left(\frac{16}{r^4} + 7\frac{f'^4}{f^4} - 20\frac{f'^3r'}{f^3r} - 4\frac{f'^2r'^2}{f^2r} + 32\frac{f'r'^3}{fr^3}\right. \\ & - 16\frac{r'^4}{r^4} - 12\frac{f'^2f''}{f^3} + 48\frac{f'r'f''}{f^2r} - 32\frac{r'^2f''}{fr^2} - 4\frac{f''^2}{f^2} - 16\frac{f'^2r''}{f^2r} + 16\frac{f'r'r''}{fr^2} + 16\frac{f''r''}{fr} \\ & \left. - 16\frac{r''^2}{r^2} + 8\frac{f'f'''}{f^2} - 16\frac{r'f'''}{fr} - 16\frac{f'r'''}{fr} + 32\frac{r'r'''}{r^2}\right)\Bigg]. \end{aligned} \quad (5)$$

The full expressions for the general ξ-dependent components of the Einstein equations are too lengthy to be shown here.

In order to solve the field equations, we must supply an appropriate set of boundary conditions. The tt and $\theta\theta$ equations are fourth-order differential equations in the two functions f and r while the ll equation is third order. This latter equation is actually a constraint which plays the role of restricting the solutions of the coupled fourth-order differential equations [21]. Since we seek wormhole solutions of (1), we shall specify the boundary data at $l = 0$, the origin of proper distance as measured from the throat. The complete set of boundary conditions therefore requires specifying eight pieces of information, namely, $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$, as well as $r(0)$, $r'(0)$, $r''(0)$, and $r'''(0)$. In particular, $r(0)$ is the (wormhole) throat radius, and we can use the ll equation to derive an exact relation between $r(0)$ and just three initial conditions. In order to do so, let us recall the boundary conditions appropriate to a wormhole (or more generally, for any spacetime with a throat). If a solution of (1) is to possess a throat, then we must require that $r(0) > 0$, $r'(0) = 0$, and $r''(0) \geq 0$. In other words, there must exist a sphere of minimum radius located at the origin of proper distance. So $r(l)$ is simply a positive increasing function of l in the neighborhood of the origin. For two-way passage through the wormhole, it is judicious to avoid solutions with horizons at the throat.

This is taken care of by requiring a (locally) nonvanishing redshift function: $f(0) > 0$ (the other possibility, $f(0) < 0$, leads to a Euclidean metric). The chain rule also provides an additional constraint, namely, $r'(0) = 0 \rightarrow f'(0) = 0$ [22]. We may ask that the redshift be a locally increasing function, $f''(0) \geq 0$. These constitute the minimum requirements. Beyond this, one could impose a symmetry on the solutions of the form $r(l) = r(-l)$ and $f(l) = f(-l)$ which automatically eliminates all the odd derivatives, $r^{(2n+1)}(0) = f^{(2n+1)}(0) = 0$. In particular, one can set $r'''(0) = f'''(0) = 0$. Taken together, these give the boundary conditions appropriate to a wormhole. However, one cannot choose freely all the eight boundary conditions independently. This is easily seen by writing out the ll equation (5) at the point $l = 0$, which yields an algebraic quartic equation for the throat radius $r(0)$,

$$-4\left(\frac{f''(0)}{f(0)}\right)^2 [1 + \ln(f(0))]r(0)^4 + 32\left(\frac{f''(0)r''(0)}{f(0)}\right)\left[1 + \frac{1}{2}\ln(f(0))\right]r(0)^3 + [K^{-2} - 16r''(0)^2\ln(f(0))]r(0)^2 + 16\ln(f(0)) = 0\,, \tag{6}$$

where we have used only that $r'(0) = f'(0) = 0$. Indeed, it is important to note that this constraining relation is *independent* of the values assigned to the third derivatives $f'''(0)$ and $r'''(0)$. With some effort, this quartic equation can be solved for $r(0)$, though the resulting expressions are not particularly transparent. However, it suffices for our purposes to consider (6) in certain simplifying but natural cases in order to get a feeling for the allowed range in $r(0)$. In this way, we have found that large throat radii can result even for values of $f(0)$, $f''(0)$, and $r''(0)$ of order unity. As case in point, taking $f(0) = f''(0) = 1$, $r''(0) = 0$, and $\xi = \frac{1}{6}$ yields $r(0) \approx 67 l_P$ where l_P is the Planck length. Larger (and smaller) throat radii are also possible depending on the values chosen for the other boundary data and the scalar coupling constant ξ.

We now turn to our specific calculations of $r(l)$ and $f(l)$ subject to the above class of boundary conditions. Employing the Rugge-Kutta method, we have obtained a good numerical solution of these equations using the following boundary conditions for $f(l)$ and $r(l)$ at the throat $l = 0$:

$$-1 \leq \ln f(0) < 0; \quad f'(0) = 0; \quad f''(0) = 0; \quad f'''(0) = 0;$$
$$r(0) = \sqrt{-16K^2 \ln f(0)}; \quad r'(0) = 0; \quad r''(0) = 0;$$
$$r'''(0) = 0\,. \tag{7}$$

The graphs which represent a particular numerical solution for $\ln f(0) = -\frac{2}{3}$ for various length scales are displayed below. The redshift function $f(l)$ is plotted versus proper distance in Figs. 1(a) and 1(b); both axes have been scaled by K^{-2}. In Fig. 1(b), we see that $f(l)$ is a positive increasing function. Actually, there is a fine structure in the form of Planck-scale "ripples" or spatial oscillations superimposed on this function; these are not numerical artifacts. These oscillations are clearly revealed in the "blow-up" graph in Fig. 1(a). The wormhole's shape function $r(l)$ is depicted in Figs. 2(a) and 2(b). In Fig. 2(a) one sees the throat, of radius $r(0) \approx 0.02 l_P$, in the neighborhood of the origin extending out to about $10K^{-2}$ at which point the wormhole "flares" out, marking the onset of the superimposed small-scale oscillations. The graph in Fig. 2(b) shows the gross features of the shape function. As remarked below, the oscillations in both $f(l)$ and $r(l)$ are composed of two different modes. Of course, since both f and r are even functions, their graphs can be reflected through the origin, $l \rightarrow -l$. The wormhole's embedding diagram is easily inferred from Figs. 2(a) and 2(b).

It is important to know the asymptotic behavior of $f(l)$ and $r(l)$. To get at this information, we carried out an asymptotic analysis of the Einstein equation (1) taking into account and guided by the results of the numerical investigation. From the numerical calculations, we see that for sufficiently large l, both the redshift and shape functions can be represented as a sum of two distinct components $f(l) = F(l) + \phi(l)$; $r(l) = R(l) + \rho(l)$, where $F(l)$ and

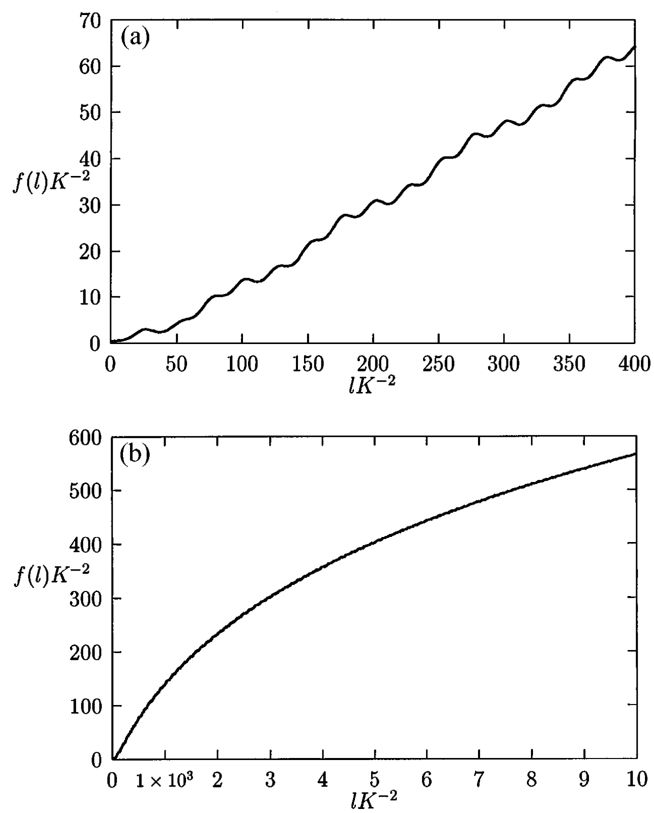

FIG. 1. (a) The redshift function on small scales. (b) The redshift function on large scales.

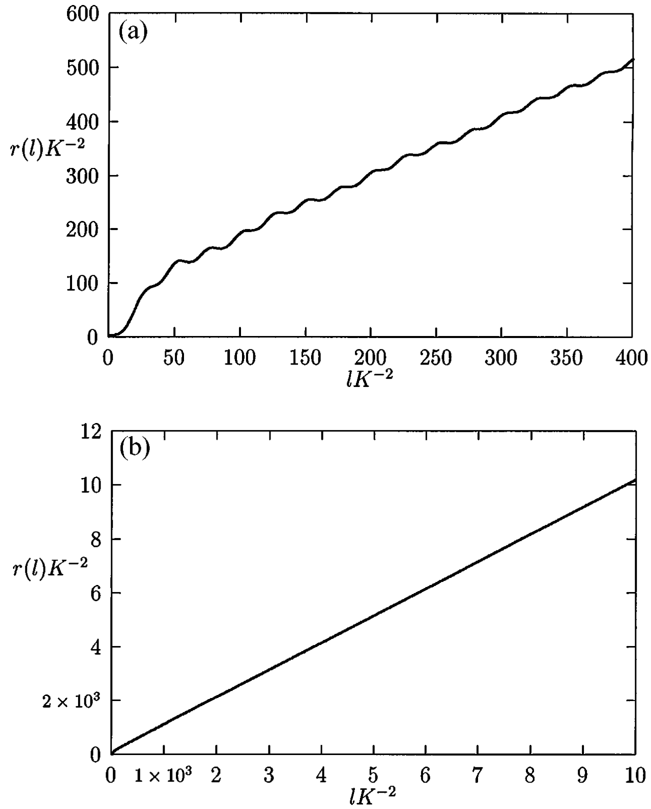

FIG. 2. (a) The wormhole shape function on small scales. (b) The wormhole shape function on large scales.

$R(l)$ are strictly *monotone increasing* and $\phi(l)$ and $\rho(l)$ are bounded *oscillating* functions.

The relative magnitudes of these components and their derivatives may be estimated straightforwardly and then used to expand consistently the coupled Einstein equations. We find that the oscillating modulation is composed of two modes with frequencies $\omega_1^2 = 1/16K^2$ and $\omega_2^2 = 1/16K^2(4 + 3\ln F)$, respectively, while in the limit of large l, $R(l) \approx l$ and $F(l) \approx (a \ln l - b)^2$ for $a = 5.3$ and $b = 25.5$.

From these combined numeric analytic calculations we see that for the chosen set of boundary conditions, $R(l) \to l$, thus our self-consistent wormhole connects two spatial regions which are asymptotically flat, modulo the Planck-scale wiggles. The redshift function, however, does not approach a constant value as $l \to \infty$, so the metric as a whole is not asymptotically flat. We have found additional self-consistent solutions of (1) by taking different values for the scalar coupling and the boundary data. In this way, we have found local solutions which correspond to large throat $[r(0) \approx 200 - 300 l_P]$ wormholes with horizons located far from the throat and wormholes connecting two bounded spatial regions. A full account of these calculations will appear in a separate publication.

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[19] We use units such that $\hbar = c = G = 1$ and our sign conventions follow those of C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[20] We have also set the scalar mass and temperature to zero: $m = \kappa = 0$. In this case, the arbitrary renormalization scale parameter μ appearing in $(T_\mu^\nu)_{\log}$ can be absorbed into the definition of f.
[21] This can be checked, for example, by writing the set of three field equations as an equivalent set of (eight) coupled first-order equations plus one algebraic equation. The radial equation reduces to the algebraic transcendental equation in f and r, i.e., it is a constraint.
[22] This follows from evaluating the identity $r'(l) \times [df(r)/dr] = f'(l)$ at $l = 0$.