Local Scale Invariance and Strongly Anisotropic Equilibrium Critical Systems

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A new set of infinitesimal transformations generalizing scale invariance for strongly anisotropic critical systems is considered. It is shown that such a generalization is possible if the anisotropy exponent $\theta = 2/N$, with $N = 1, 2, 3 \ldots$. Differential equations for the two-point function are derived and explicitly solved for all values of N. Known special cases are conformal invariance ($N = 2$) and Schrödinger invariance ($N = 1$). For $N = 4$ and $N = 6$, the results contain as special cases the exactly known scaling forms obtained for the spin-spin correlation function in the axial next-nearest-neighbor spherical model at its Lifshitz points of first and second order. [S0031-9007(97)02617-3]

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The notion of scale invariance is central to the present understanding of critical phenomena. Here we are interested in strongly anisotropic criticality. There are many physical examples of this, like critical dynamics and nonequilibrium dynamics [1], domain growth [2], magnetic systems with competing interactions [3], or particle reaction systems such as directed percolation. By definition, these systems are characterized by the condition that the critical two-point functions C transform under rescaling as

$$C(br, b^{\theta}t) = b^{-2x}C(r,t), \tag{1}$$

where r, t label "space" and "time" coordinates, x is a scaling dimension, and $\theta = \nu_{\|}/\nu_{\perp}$ is the anisotropy exponent (in many cases, it is also referred to as the dynamic exponent z). In this Letter, we confine ourselves to strongly anisotropic *equilibrium* systems.

Equation (1) can be rewritten as

$$C(r,t) = t^{-2x/\theta}\Phi\left(\frac{r^{\theta}}{t}\right), \tag{2}$$

where $\Phi(u)$ is a scaling function. Some information on the form of $\Phi(u)$ is readily available. For $r = 0$, one expects $C(0,t) \sim t^{-2x/\theta}$ and for $t = 0$, one expects $C(r,0) \sim r^{-2x}$. This implies $\Phi(u) \simeq \Phi_0$ for $u \to 0$ and $\Phi(u) \simeq \Phi_{\infty}u^{-2x/\theta}$ for $u \to \infty$, where $\Phi_{0,\infty}$ are generically nonvanishing constants.

Is it possible to obtain more information about $\Phi(u)$ on a general basis without going back to explicit model calculations?

Indeed, this has been affirmatively answered in two cases. First, for *isotropic* critical systems, that is, for $\theta = 1$, the extension of Eq. (1) to space-dependent rescaling factors $b = b(\vec{r})$ leads to the requirement of *conformal invariance* of the correlation functions [4]. (We are not going to restrict ourselves to two dimensions and shall thus sidestep the extremely powerful and elegant work done in 2D, as initiated in Ref. [5].) Then the critical two-point correlation function is, up to normalization [4]

$$C(\vec{r}) = \langle\phi_1(\vec{r}_1)\phi_2(\vec{r}_2)\rangle = \delta_{x_1,x_2}|\vec{r}_1 - \vec{r}_2|^{-2x_1}, \tag{3}$$

where $x_{1,2}$ are the scaling dimensions of the (scalar) fields $\phi_{1,2}$ which are assumed to be quasiprimary in the sense of Ref. [5].

Second, for $\theta = 2$, the extension of Eq. (1) to space-time-dependent scaling $b = b(\vec{r}, t)$ leads to the requirement of *Schrödinger invariance* [6,7]. Since this corresponds to the "nonrelativistic" limit of the conformal group [8], local fields ϕ_i are characterized by two quantum numbers, the scaling dimensions x_i, and the masses $\mathcal{M}_i \geq 0$. For scalar quasiprimary fields, the two-point function is, up to normalization [9,10]

$$\langle\phi_1(\vec{r}_1,t_1)\phi_2^*(\vec{r}_2,t_2)\rangle = \delta_{x_1,x_2}(t_1 - t_2)^{-x_1}\delta_{\mathcal{M}_1,\mathcal{M}_2} \times \exp\left(-\frac{\mathcal{M}_1}{2}\frac{(\vec{r}_1 - \vec{r}_2)^2}{t_1 - t_2}\right) \tag{4}$$

with $t_1 > t_2$. In comparing Eqs. (3) and (4), we note that the first line of (4) is similar to the conformal invariance result, while the terms containing the masses reflect the nonrelativistic nature of the problem for $\theta = 2$. For $\theta = 1$, Eq. (3) is completely standard and there are quite a few statistical mechanics models with $\theta = 2$ which reproduce (4); see Refs. [10,11].

What are common features of conformal and Schrödinger transformations which might serve as a basis for generalizing beyond $\theta = 1, 2$? For notational simplicity, we shall work from now on in two space dimensions or one time and one space dimension, respectively, but the generalization to any number of dimensions is immediate. Working in (complex) light-cone coordinates $z = x + iy, \bar{z} = x - iy$, the conformal transformations are

$$z \to z' = \frac{\alpha z + \beta}{\gamma z + \delta}; \qquad \alpha\delta - \beta\gamma = 1, \tag{5}$$

and similarly for $\bar{z}$. The infinitesimal generators are $\ell_n = -z^{n+1}\partial_z$ and satisfy the commutation relations $[\ell_n, \ell_m] = (n - m)\ell_{n+m}$. The set $\{\ell_{\pm 1}, \ell_0\}$ generates the Möbius transformations (5). The space-time transformations of the Schrödinger group are [6,7]

$$t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad r \to r' = \frac{r + vt + a}{\gamma t + \delta}, \tag{6}$$

(with $\alpha\delta - \beta\gamma = 1$) which contains the Galilei group as a subgroup. As is well known from nonrelativistic quantum mechanics, the wave function $\psi(r,t)$ transforms under a unitary *projective* representation $\mathcal{U}$ of the Galilei transformation [12]

$$\mathcal{U}^{-1}\psi(r,t)\mathcal{U} = \exp\left[\frac{im}{2}(v^2 t + 2vr)\right]\psi(r + vt, t), \tag{7}$$

where $m \geq 0$ is the mass of the particle. This gives rise to the Bargmann superselection rule [12,7] already present in (4). If a wave function ψ is characterized by the mass $m \geq 0$, its complex conjugate ψ^* is characterized by $-m$. This correspondence between a field ϕ and ϕ^* is to be kept when going over to diffusive behavior $m \to i\mathcal{M}$. An analogous statement applies to the full Schrödinger group [7,13]. The infinitesimal generators must therefore contain mass terms and may be written in the form [10]

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2,$$
$$Y_m = -t^{m+1/2}\partial_r - \left(m + \frac{1}{2}\right)\mathcal{M}t^{m-1/2}r, \tag{8}$$
$$M_n = -t^n\mathcal{M}.$$

and the nonvanishing commutators are

$$[X_n, X_m] = (n - m)X_{n+m},$$
$$[X_n, Y_m] = \left(\frac{n}{2} - m\right)Y_{n+m},$$
$$[X_n, M_m] = -mM_{n+m}, \qquad [Y_n, Y_m] = (n - m)M_{n+m}.$$

The set $\{X_{\pm1}, X_0, Y_{\pm1/2}, M_0\}$ generates the transformations (6).

We now specify the conditions under which we shall attempt to consider an arbitrary value of the exponent θ. These conditions are formulated as to remain as close as possible to the known situations of either conformal or Schrödinger invariance.

(1) Since in both cases, Möbius transformations play a prominent role, we shall seek space-time transformations which in the time coordinate undergoes a Möbius transformation

$$t \to t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \qquad \alpha\delta - \beta\gamma = 1. \tag{9}$$

(2) The generator for scale transformations should read $X_0 = -t\partial_t - \frac{1}{\theta}r\partial_r$.

(3) Spatial translation invariance is required.

(4) The generators should contain "mass" terms, built in analogy to the mass terms for $\theta = 2$ in (8).

(5) We want to use these transformations to derive differential equations for the two-point functions. We shall require that when applied to a two-point function, the generators will yield a *finite* number of independent conditions. Thus the operators applied to the two-point functions should provide a realization of a finite-dimensional Lie algebra.

We now proceed to list the consequences of the above assumptions. The generator X_n, $n = -1, 0, 1$ of the Möbius transformations must contain the term $X_n = -t^{n+1}\partial_t + \ldots$ and thus satisfy the commutation relations $[X_n, X_m] = (n - m)X_{n+m}$. In order to keep the "conformal" structure of the transformations, we must require that these commutation relations are also satisfied by the final generators X_n. Then the explicit form of X_0 implies that up to mass terms, $X_n = -t^{n+1}\partial_t - \theta^{-1}(n+1)t^n r\partial_r$. Next, we study the action of X_n on the space translation operator $-\partial_r$. We shall write $\theta = 2/N$ and define, up to mass terms, the operators $Y_m = -t^{N/2+m}\partial_r$ with $m = -N/2 = k$, $k = 0, 1, \ldots$. The nonvanishing commutators of X_n and Y_m are

$$[X_n, X_m] = (n - m)X_{n+m},$$
$$[X_n, Y_m] = \left(N\frac{n}{2} - m\right)Y_{n+m}, \tag{10}$$

In particular, $[X_1, Y_{-N/2+k}] = (N - k)Y_{-N/2+k+1}$. Thus, the repeated action of X_1 on $Y_{-N/2} = -\partial_r$ is creating an infinite set of generators. This can be truncated only if $N = 2/\theta$ is a positive integer, $N = 1, 2, \ldots$. Therefore the list of possible values of θ is

$$\theta = \frac{2}{N} = 2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \ldots. \tag{11}$$

A few remarks are in order. The conformal properties of the tranformations sit in the time direction. It should thus be the temporal degrees of freedom which render the system critical. Therefore one should expect that the results for the two-point function to be derived below should apply independently of whether or not the "spatial" degrees of freedom by themselves furnish a critical system. One might think of interchanging the roles of space and time coordinates and thus obtain a set of anisotropy exponents $\theta = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. To do this, however, one must impose conformal invariance on the spatial degrees of freedom and this means that the spatial degrees of freedom alone should describe a system at a critical point. While that would be fine for a study of critical dynamics, many other examples of strongly anisotropic critical systems are not at a static critical point. In $(1+1)D$, however, this distinction should not be very important, since a one-dimensional subsystem with short-ranged interactions cannot order by itself.

Finally, we have to see whether it is possible to include mass terms into the generators X_n, Y_m without spoiling the commutator relations (10). Indeed, this can be done. The details of this calculation will be presented elsewhere; here we merely quote the result. One solution for the generators X_1 and $Y_{-N/2+1}$ (which generate the so-called "special" and "Galilei" transformations) is

$$X_1 = -t^2\partial_t - Ntr\partial_r - \alpha r^2\partial_t^{N-1},$$
$$Y_{-N/2+1} = -t\partial_r - \frac{2\alpha}{N} r\partial_t^{N-1}, \tag{12}$$

where α is a dimensionful, in general nonuniversal, constant which parametrizes the mass term. When applying these generators to a two-point function $C = \langle\phi_1\phi_2\rangle$, where the fields are characterized by two quantum numbers, the scaling dimension x_i and the "mass" α_i, consistency can be achieved only if

$$\alpha_1 = (-1)^N\alpha_2 . \tag{13}$$

We point out that for systems with N even, the distinction between ϕ and ϕ^* becomes unnessary. In principle, it is even possible to introduce a *universal* mass constant α which is the same for all fields. On the other hand, for N odd, the α_i must be kept as peculiar quantum numbers of the fields ϕ_i. To each field ϕ_i, characterized by the numbers (x_i, α_i), there is a conjugate field ϕ_i^* characterized by $(x_i, -\alpha_i)$. Furthermore, it can be checked using (13) that the two-particle operators built from the X_n, Y_m provide on C a realization of the Lie algebra (10).

Two special cases can be easily recognized. For $N = 2$, we recover the familiar conformal algebra, with $X_n = \ell_n + \overline{\ell}_n$ and $Y_n = i(\ell_n - \overline{\ell}_n)$, $n = -1, 0, 1$, provided that the mass $\alpha = -c^{-2}$ (where c is the speed of light, normally set to $c = 1$ when introducing light-cone coordinates $z, \overline{z} = t \pm \sqrt{\alpha}\, r$). For $N = 1$, we recover the generators (8) of the Schrödinger algebra, with $\alpha_i = \frac{1}{2}\mathcal{M}_i$.

We are now ready to calculate the two-point function explicitly. If $X_n^{(a)}$ is the generator X_n acting on particle a, $a = 1, 2$ (and similarly for the Y_m), the two-particle operators are $\widetilde{X}_n = X_n^{(1)} + X_n^{(2)}$. We are interested in the two-point function

$$G(r_1, r_2; t_1, t_2) = \langle\phi_1(r_1, t_1)\phi_2^*(r_2, t_2)\rangle, \tag{14}$$

and the covariance of G is expressed through the conditions (meaning that the ϕ_i are quasiprimary [5])

$$\widetilde{X}_0 G = \frac{x_1 + x_2}{\theta} G,$$
$$\widetilde{X}_1 G = \left(\frac{x_1}{\theta} t_1 + \frac{x_2}{\theta} t_2\right) G, \tag{15}$$
$$\widetilde{X}_{-1} G = \widetilde{Y}_m G = 0,$$

with $m = -N/2, -N/2 + 1, \ldots, N/2$. We write $t = t_1 - t_2$ and $r = r_1 - r_2$. In addition, we put $\zeta = (x_1 + x_2)/\theta$. The scaling of the two-point function can be written as

$$G = G(r, t) = \delta_{x_1,x_2}\delta_{\alpha_1,\alpha_2} r^{-2x_1}\Omega\left(\frac{t}{r^{2/N}}\right), \tag{16}$$

where $\Omega(v)$ satisfies the differential equation

$$\alpha_1\Omega^{(N-1)}(v) - v^2\Omega'(v) - \zeta v\Omega(v) = 0 \tag{17}$$

subject to the boundary conditions $\Omega(0) = \text{const}$ and $\Omega(v) \sim v^{-\zeta}$ as $v \to \infty$. The general solution (for $N \geq 2$) of Eq. (17) is

$$\Omega(v) = \sum_{p=0}^{N-2} b_p v^p \mathcal{F}_p ;$$
$$\mathcal{F}_p = {}_2F_{N-1}\left(\frac{\zeta + p}{N}, 1; 1 + \frac{p}{N}, 1 + \frac{p-1}{N}, \ldots, \frac{p+2}{N}; \frac{v^N}{N^{N-2}\alpha_1}\right), \tag{18}$$

where ${}_2F_{N-1}$ is a generalized hypergeometric function and the b_p are free parameters. In order to check the boundary conditions, we recall the known [14] asymptotic behavior of the $\mathcal{F}_p$. The leading behavior for $v \to \infty$ for each term is of the order $\exp[A(N-2)v^{N/(N-2)}]$, where the constant $A > 0$. For $N \geq 3$ the condition

$$\sum_{p=0}^{N-2} b_p \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{N})\Gamma(\frac{p+\zeta}{N})}\left(\frac{\alpha_1}{N^2}\right)^{p/N} = 0 \tag{19}$$

is sufficient to cancel the entire exponential contribution. Eliminating b_{N-2}, the final result becomes $\Omega(v) = \sum_{p=0}^{N-3} b_p\Omega_p(v)$, with $b_0 \neq 0$. The asymptotic behaviour

$$\Omega_p(v) \cong \begin{cases} v^p; & v \to 0, \\ \Omega_\infty v^{-\zeta}; & v \to \infty \end{cases} \tag{20}$$

is found to be in complete agreement with the requested boundary conditions, where

$$\Omega_p(v) = v^p\mathcal{F}_p - \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{N})\Gamma(\frac{p+\zeta}{N})} \frac{\Gamma(\frac{N-1}{N})\Gamma(1 + \frac{\zeta-2}{N})}{\Gamma(N-1)} \times \left(\frac{\alpha_1}{N^2}\right)^{(p+2-N)/N} v^{N-2}\mathcal{F}_{N-2}, \tag{21}$$

$$\Omega_\infty = -\left(\frac{\alpha_1}{N^2}\right)^{(\zeta+p)/N} \frac{\Gamma(\frac{1-\zeta}{N})}{\Gamma(1-\zeta)} \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{N})} \times \frac{\pi\sin[\frac{\pi}{N}(p+2)]}{\Gamma(\frac{p+\zeta}{N})\sin[\frac{\pi}{N}(p+\zeta)]\sin[\frac{\pi}{N}(\zeta-2)]}. \tag{22}$$

Equation (16) together with Eqs. (18) and (19) or (21) gives the solution to our question. After normalization, $N - 3$ of the parameters b_p are still arbitrary.

It remains to be seen whether there exist examples which do reproduce these predictions. Here, we shall consider the spin-spin correlator in spin systems with axial next nearest neighbor interactions [15,3]. The spin Hamiltonian is

$$\mathcal{H} = -J{\sum_{(i,j)}}' s_i s_j + \kappa J\sum_{i\|} s_{i\|}s_{i\|+1}, \tag{23}$$

where s_i is a $O(n)$ vector spin and the first term ($J > 0$) describes nearest-neighbor ferromagnetic interactions while the second term ($\kappa > 0$) contains next-nearest-neighbor interactions along a single axis. By definition [15], the meeting point of the paramagnetic, ferromagnetic, and incommensurable phases of the model is termed a *Lifshitz point* (of first order) and is known to show strongly anisotropic scaling, with correlation length exponents $\nu_\| = \nu_{\ell 4}$, $\nu_\perp = \nu_{\ell 2}$ measured parallel and perpendicular to the axis. The anisotropy exponent $\theta = \nu_\| / \nu_\perp = 1/2$ independently [15] of the value of n. This corresponds to $N = 4$. The fact that $\theta = \frac{1}{2}$ stays fixed at its mean-field value may point toward the existence of a hidden symmetry which prevents its renormalization [16].

In the $n \to \infty$ limit one recovers the spherical (or ANNNS [3]) model and the spin-spin correlation function $C(r_\|, \vec{r}_\perp) = \langle s_{r_\|, \vec{r}_\perp} s_{0, \vec{0}} \rangle$ *at* the Lifshitz point is exactly known in d dimensions. The result is [17]

$$C(r_\|, \vec{r}_\perp) = C_0 r_\perp^{-(d-d_*)} \Psi\left(\frac{d - d_*}{2}, \sqrt{\frac{1}{32 c_2}} \frac{r_\|^2}{r_\perp} \right), \quad (24)$$

where C_0 and c_2 are known (nonuniversal) constants, d_* is the lower critical dimension, and $\Psi(a, x) = \sum_{k=0}^\infty \frac{(-x)^k}{k!} \frac{\Gamma(k/2 + a)}{\Gamma(k/2 + 3/4)}$. On the other hand, for $N = 4$ Eq. (16) gives $G(r, t) \sim r^{-\zeta/2} \Omega(v)$. As for the scaling function $\Omega(v)$, we have from (21) that for $N = 4$

$$\Omega_0(v) = \frac{\Gamma(3/4)}{\Gamma(\zeta/4)} \Psi\left(\frac{\zeta}{4}, \frac{v^2}{2\sqrt{\alpha_1}} \right). \quad (25)$$

Thus, with the correspondence $t \leftrightarrow r_\|$, $r \leftrightarrow r_\perp$, and $\alpha_1 = 8 c_2$, the order parameter scaling function for the ANNNS model *at* the first order Lifshitz point is exactly reproduced for the parameter value $b_1 = 0$.

Higher order Lifshitz points [3] can be reached by adding further axial interaction terms in (23). Second order Lifshitz points correspond to $\theta = \frac{1}{3}$ or $N = 6$. We have checked that the exactly known spin-spin correlation function for the ANNNS model [17] does agree with the scaling form (21).

A tempting open question is whether the scaling function of the spin-spin correlator of the ANNNI model at the Lifshitz point (in 3D [3]), which still corresponds to $N = 4$ [15], can be described in the same framework with a different value of b_1. Recently, a new asymmetric six-vertex model with a $\theta = \frac{1}{2}$ critical point has been described [18]. Further examples might be provided by the superintegrable chiral N-state Potts model, where [19] $\nu_\tau = 2/N$, $\nu_x = 1$ at the self-dual point or else by a non-Hermitian quantum chain obtained from the asymmetric clock model, where [20] $\nu_x = 0.95(4)$ and $\nu_\tau = 0.67(4)$. The possibility of applying the above scheme to the Kardar-Parisi-Zhang (KPZ) equation [1], which in $(1 + 1)D$ has $\theta = \frac{3}{2}$, seems worth exploring [21]. Finally, it appears possible to extend the present approach to yield the scaling forms for the response functions out of equilibrium (as already checked [10] in a few cases for Schrödinger invariance) and to higher n-point functions. This will be reported elsewhere. All in all, further explicit model results will be needed in order to gauge the merits of this or any other general approach to strongly anisotropic scaling.

In conclusion, we have examined a set of infinitesimal transformations which for $\theta = 2/N$, $N = 1, 2, 3, \ldots$ generalize scale invariance. We have seen how to calculate from these the two-point functions for strongly anisotropic equilibrium critical systems. Lifshitz points in the ANNNS (spherical) model apparently provide model examples which realize these transformations.

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