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Quantum Jumps as Decoherent Histories

Todd A. Brun*
Department of Physics, Queen Mary and Westfield College, University of London, London E1 4NS, England
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Quantum open systems are described in the Markovian limit by master equations in Lindblad form. I argue that common "quantum jump" techniques, which solve the master equation by unraveling its evolution into stochastic trajectories in Hilbert space, correspond closely to a particular set of decoherent histories. This is illustrated by a simple model of a photon counting experiment. [S0031-9007(97)02717-8]

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Recently a great deal of work has been done in quantum optics on "quantum jump" simulations of continuously measured systems with dissipation [1–5]. In this technique, a system described by a master equation for the reduced density operator ρ in the Markovian approximation [6],

$$\dot{\rho} = -i[\hat{H}, \rho] + \sum_m \hat{L}_m \rho \hat{L}_m^\dagger - \frac{1}{2} \hat{L}_m^\dagger \hat{L}_m \rho - \frac{1}{2} \rho \hat{L}_m^\dagger \hat{L}_m , \tag{1}$$

is "unraveled" into a jump process for pure states. $\hat{H}$ is the system Hamiltonian, and the $\{\hat{L}_m\}$ are a set of *Lindblad operators* which model the effects of the environment.

Around the same time, the decoherent histories formulation of quantum mechanics was developed [7–12]. In this formalism, one describes a quantum system in terms of an exhaustive set of possible histories, which must satisfy a *decoherence* or *consistency* criterion. Histories which satisfy this criterion have probabilities which obey the usual classical probability sum rules.

Both quantum trajectories and decoherent histories describe a quantum system in terms of alternative possible evolutions; they thus bear a certain resemblance to each other. What is more, quantum jumps are commonly interpreted as giving the results of continuous measurements, and histories which correspond to records of a "classical" measuring device should always decohere [10]. Thus, there should be a set of decoherent histories which corresponds to the quantum trajectories of a continuously measured system.

Exactly such a correspondence has recently been shown between decoherent histories and quantum state diffusion (QSD), another unraveling of the master equation, by Diósi *et al.* [13]. Though this result was pioneering, it was rather abstract, and lacked any direct connection to a physical measurement situation. Similar results for yet another unraveling were given by Paz and Zurek [14] and Diósi [15] in a model with exact decoherence, but also far removed from physical measurement situations. Other treatments [16] have been framed in terms of measurement alone.

Consider a quantum system with a Hamiltonian $\hat{H}_0$, completely isolated except for a single channel of decay, which is monitored by an external photon detector. We model this detector as a single two-level system (the "output mode") with states $|0\rangle$ and $|1\rangle$ strongly coupled to an environment representing the remaining degrees of freedom of the device.

The measuring device produces two important effects. The first is dissipation. Excitations of the output mode will be absorbed by the measuring device with a rate Γ_1 which we assume to be rapid compared to the dynamical time scale of the system. The time $1/\Gamma_1$ represents the time resolution of the detector.

The second effect is more subtle but just as important: decoherence. As the state of the output mode becomes correlated with the internal degrees of freedom of the measuring device, the phase coherence between the ground and excited states of the output mode is lost. Investigations of this process have shown that the loss of coherence is generally far quicker than the actual rate of energy loss [17]. This decoherence rate is $\Gamma_2 \gg \Gamma_1$.

We suppose that the system is linearly coupled to the output mode via the Hamiltonian,

$$\hat{H}_I = \kappa(\hat{a}^\dagger \otimes \hat{b} + \hat{a} \otimes \hat{b}^\dagger), \quad (2)$$

and the total Hamiltonian is

$$\hat{H} = \hat{H}_0 \otimes \hat{1} + \kappa(\hat{a}^\dagger \otimes \hat{b} + \hat{a} \otimes \hat{b}^\dagger), \quad (3)$$

where $\hat{a}$ and $\hat{b}$ ($\hat{a}^\dagger$ and $\hat{b}^\dagger$) are the lowering (raising) operators for the system and output mode, respectively. The hierarchy of evolution rates is $\Gamma_2 \gg \Gamma_1 \gg \kappa$.

The total system obeys the master equation,

$$\dot{\rho} = -i[\hat{H},\rho] + \Gamma_1 \hat{b}\rho\hat{b}^\dagger - \frac{\Gamma_1}{2}\hat{b}^\dagger\hat{b}\rho - \frac{\Gamma_1}{2}\rho\hat{b}^\dagger\hat{b} + \Gamma_2\sigma_z\rho\sigma_z - \Gamma_2\rho \equiv \mathcal{L}\,\rho, \quad (4)$$

where ρ is the density matrix for the combined system and output mode, and the Pauli operator σ_z acts on the output mode. $\mathcal{L}$ is the *Liouville superoperator*. This is a linear equation, and so can be formally solved:

$$\rho(t_2) = \exp[\mathcal{L}(t_2 - t_1)]\rho(t_1). \quad (5)$$

Assume that we start in a pure state $|\Psi\rangle = |\psi\rangle \otimes |0\rangle$. We can expand ρ,

$$\rho(t) = \rho_{00}(t) \otimes |0\rangle\langle 0| + \rho_{01}(t) \otimes |0\rangle\langle 1| + \rho_{10}(t) \otimes |1\rangle\langle 0| + \rho_{11}(t) \otimes |1\rangle\langle 1|. \quad (6)$$

In terms of these components the master equation becomes

$$\begin{aligned}
\dot{\rho}_{00} &= -i[\hat{H}_0,\rho_{00}] - i\kappa\hat{a}^\dagger\rho_{10} + i\kappa\rho_{01}\hat{a} + \Gamma_1\rho_{11}, \\
\dot{\rho}_{01} &= -i[\hat{H}_0,\rho_{01}] - i\kappa\hat{a}^\dagger\rho_{11} + i\kappa\rho_{00}\hat{a}^\dagger - G\rho_{01} \\
&= \dot{\rho}_{10}^\dagger, \\
\dot{\rho}_{11} &= -i[\hat{H}_0,\rho_{11}] - i\kappa\hat{a}\rho_{01} + i\kappa\rho_{10}\hat{a}^\dagger - \Gamma_1\rho_{11}, \quad (7)
\end{aligned}$$

where $G = \Gamma_1/2 + 2\Gamma_2 \gg \Gamma_1 \gg \kappa$. (This combination G occurs frequently in the equations which follow.)

Since the $\rho_{01}, \rho_{10}, \rho_{11}$ components are heavily damped, we can adiabatically eliminate all components other than ρ_{00} [18]:

$$\dot{\rho}_{00} = -i[\hat{H}_0,\rho_{00}] + \frac{2\kappa^2}{G}\hat{a}\rho_{00}\hat{a}^\dagger - \frac{\kappa^2}{G}\hat{a}^\dagger\hat{a}\rho_{00} - \frac{\kappa^2}{G}\rho_{00}\hat{a}^\dagger\hat{a}, \quad (8)$$

to first order in κ^2/G, provided that the system is not so highly excited as to emit too rapidly, i.e., $\kappa\langle\hat{a}^\dagger\hat{a}\rangle \ll \Gamma_1$.

We can unravel the master equation (8) into a sum over quantum jump trajectories. First, define a non-Hermitian *effective Hamiltonian,*

$$\hat{H}_{\text{eff}} = \hat{H}_0 - i(\kappa^2/G)\hat{a}^\dagger\hat{a}. \quad (9)$$

Assume that the system (excluding the output mode) begins in a pure state $|\psi\rangle$. $|\psi\rangle$ evolves according to the Schrödinger equation,

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}\hat{H}_{\text{eff}}|\psi\rangle, \quad (10)$$

interrupted at random times by sudden quantum jumps

$$|\psi\rangle \to \hat{a}|\psi\rangle. \quad (11)$$

These jumps correspond to the detection of photons [1,2]. Note that this evolution does not preserve the norm of the state. The physical state is taken to be $|\tilde{\psi}\rangle = |\psi\rangle/\sqrt{\langle\psi\,|\,\psi\rangle}$, the renormalized state.

The probability that an initial state $|\psi\rangle$ evolves for a time T and undergoes N jumps during intervals δt centered at times $t_1,\ldots,t_N$ is

$$(2\delta t\kappa^2/G)^N\,\text{Tr}\{e^{-i\hat{H}_{\text{eff}}(T-t_N)}\hat{a}e^{-i\hat{H}_{\text{eff}}(t_N-t_{N-1})}\hat{a}\cdots\hat{a}e^{-i\hat{H}_{\text{eff}}t_1} \times |\psi\rangle\langle\psi|e^{i\hat{H}^\dagger_{\text{eff}}t_1}\hat{a}^\dagger\cdots\hat{a}^\dagger e^{i\hat{H}^\dagger_{\text{eff}}(T-t_N)}\}, \quad (12)$$

i.e., the norm of the unrenormalized state gives the probability for that state to be realized.

Equation (8) is valid only as long as the Markovian approximation remains good. In the case of our toy model, this means that it is valid only on time scales longer than $1/\Gamma_1$. Thus, rather than a jump occurring at a time t_i, it is more correct to consider the jump as occurring during an interval $\delta t \sim 1/\Gamma_1$ centered on t_i. This is fine as long as the jumps are separated by more than δt on average, i.e., the system is not too highly excited.

By averaging $|\tilde{\psi}\rangle\langle\tilde{\psi}|$ over all possible trajectories with the probability measure (12), one can show that this unraveling reproduces the master equation (8) as required [3].

Now, let us turn to the decoherent histories picture. In nonrelativistic quantum mechanics, a set of histories for a system can be specified by choosing a complete set of projections $\{\hat{\mathcal{P}}^j_{\alpha j}(t_j)\}$ at a sequence of times $t_1,\ldots,t_N$, which represent different exclusive alternatives:

$$\sum_{\alpha_j}\hat{\mathcal{P}}^j_{\alpha_j}(t_j) = \hat{1}, \qquad \hat{\mathcal{P}}^j_{\alpha_j}(t_j)\hat{\mathcal{P}}^j_{\alpha'_j}(t_j) = \delta_{\alpha_j\alpha'_j}\hat{\mathcal{P}}^j_{\alpha_j}(t_j). \quad (13)$$

A particular history (denoted h) is given by choosing one $\hat{\mathcal{P}}$ at each point in time. The *decoherence functional* on a pair of histories h and h' is

$$D[h,h'] = \text{Tr}\{\hat{\mathcal{P}}^N_{\alpha_N}(t_N)\cdots\hat{\mathcal{P}}^1_{\alpha_1}(t_1)\rho(t_0)\hat{\mathcal{P}}^1_{\alpha'_1}(t_1)\cdots\hat{\mathcal{P}}^N_{\alpha'_N}\}, \quad (14)$$

where $\rho(t_0)$ is the initial density matrix of the system [10]. This satisfies the *decoherence criterion* if the off-diagonal terms vanish, $D[h,h'] = 0, h \neq h'$. The diagonal terms then give the probabilities of the histories, $p(h) = D[h,h]$.

Suppose our initial pure state is $|\Psi\rangle = |\psi_0\rangle \otimes |0\rangle$, and we consider histories composed only of the Schrödinger projections,

$$\hat{\mathcal{P}}_0 = \hat{1} \otimes |0\rangle\langle 0|, \qquad \hat{\mathcal{P}}_1 = \hat{1} \otimes |1\rangle\langle 1|, \tag{15}$$

representing the absence or presence of a photon in the output mode. The projections are spaced a short time δt apart, and a history is composed of N projections, representing a total time $T = N\delta t$. A single history h is specified by a string $\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, where $\alpha_j = 0, 1$. In this case, by the quantum regression theorem [19] the decoherence functional (14) becomes

$$D[h,h'] = \mathrm{Tr}\{\hat{\mathcal{P}}_{\alpha_N} e^{\mathcal{L}\,\delta t}(\hat{\mathcal{P}}_{\alpha_{N-1}} e^{\mathcal{L}\,\delta t} \times (\cdots e^{\mathcal{L}\,\delta t}(\hat{\mathcal{P}}_{\alpha_1}|\Psi\rangle\langle\Psi|\hat{\mathcal{P}}_{\alpha_1'})\cdots)\hat{\mathcal{P}}_{\alpha_N'})\}. \tag{16}$$

The Liouville time evolution superoperators (5) evolve pure states into mixed states. This is counteracted by the effect of the repeated projections $\hat{\mathcal{P}}_\alpha$.

From Eq. (7), we can determine the character of the different histories. The crucial parameter is the size of the spacing δt between projections. The interesting regime is in the range

$$\frac{1}{G} \ll \delta t \ll \frac{1}{\Gamma_1}. \tag{17}$$

On this time scale, the Γ_2 terms are sufficient to ensure decoherence, while the effects of the Γ_1 terms are resolved into individual pure state trajectories. This last is a subtle point. The probability of a photon being emitted in any single time step is small. However, if a photon *is* emitted, it has an appreciable possibility of being absorbed on a time scale $1/\Gamma_1$. The effect of decoherence produces the terms $(\kappa^2/G)\hat{a}^\dagger\hat{a}\rho_{00}$ and $(\kappa^2/G)\rho_{00}\hat{a}^\dagger\hat{a}$ in Eq. (8), which are included in the effective Hamiltonian (9). These terms are already important on a time scale $\delta t \ll 1/\Gamma_1$. By contrast, the term $(2\kappa^2/G)\hat{a}\rho_{00}\hat{a}^\dagger$ is produced by the effects of dissipation, which only become important on a time scale $1/\Gamma_1$. It is this term which causes pure states to evolve into mixed states in Eq. (8). By choosing a time $\delta t \ll 1/\Gamma_1$, we can maintain the purity of the system state over a full trajectory, as we shall see.

If the external mode is initially unexcited, with $\rho = \rho_{00} \otimes |0\rangle\langle 0|$, then after evolving for a time δt the state becomes

$$\begin{aligned}(e^{\mathcal{L}\,\delta t}\rho)_{00} &= \rho_{00} - i[\hat{H}_0, \rho_{00}]\delta t - \frac{\kappa^2}{G}\hat{a}^\dagger\hat{a}\rho_{00}\delta t \\ &\quad - \frac{\kappa^2}{G}\rho_{00}\hat{a}^\dagger\hat{a}\delta t + \text{h.o.t.} \\ &\approx e^{-i(\hat{H}_0 - i(\kappa^2/G)\hat{a}^\dagger\hat{a})\delta t}\rho_{00}e^{i(\hat{H}_0 + i(\kappa^2/G)\hat{a}^\dagger\hat{a})\delta t}, \\ (e^{\mathcal{L}\,\delta t}\rho)_{01} &= \frac{i\kappa}{G}\rho_{00}\hat{a}^\dagger + \text{h.o.t.} = (e^{\mathcal{L}\,\delta t}\rho)_{10}^\dagger, \\ (e^{\mathcal{L}\,\delta t}\rho)_{11} &= \frac{2\kappa^2}{G}\hat{a}\rho_{00}\hat{a}^\dagger\delta t + \text{h.o.t.}\end{aligned} \tag{18}$$

Here we see the appearance of the effective Hamiltonian $\hat{H}_{\mathrm{eff}}$, just as in the quantum jump unraveling.

If the initial state is $\rho = \rho_{11} \otimes |1\rangle\langle 1|$, after a time δt the state becomes

$$\begin{aligned}(e^{\mathcal{L}\,\delta t}\rho)_{00} &= \Gamma_1\delta t e^{-i\hat{H}_{\mathrm{eff}}\delta t}\rho_{11}e^{i\hat{H}_{\mathrm{eff}}^\dagger\delta t} + \frac{2\kappa^2}{G}\hat{a}^\dagger\rho_{11}\hat{a} \\ &\quad + \text{h.o.t.}, \\ (e^{\mathcal{L}\,\delta t}\rho)_{01} &= -\frac{i\kappa}{G}\hat{a}^\dagger\rho_{11} + \text{h.o.t.} = (e^{\mathcal{L}\,\delta t}\rho)_{10}^\dagger, \\ (e^{\mathcal{L}\,\delta t}\rho)_{11} &= (1 - (\Gamma_1 + 2\kappa^2/G)\delta t)e^{-i\hat{H}_{\mathrm{eff}}\delta t}\rho_{11}e^{i\hat{H}_{\mathrm{eff}}^\dagger\delta t} \\ &\quad + \text{h.o.t.},\end{aligned} \tag{19}$$

Once again the effective Hamiltonian appears, together with two additional effects. The first is the possibility that the photon in the excited mode will be absorbed by the measuring device. The second (much smaller) effect is the possibility that the photon will be coherently reabsorbed by the system. This last process is so weak as to be negligible.

By combining the above expressions with the appropriate projections $\hat{\mathcal{P}}_0$ and $\hat{\mathcal{P}}_1$ (which pick out the ρ_{00} or ρ_{11} component, respectively), we can write down the probabilities of all possible histories.

Note that the magnitude of the off-diagonal $\rho_{01,10}$ terms in both cases is of order $O(\kappa/G)$. (This is also true for transitions from off-diagonal to diagonal terms.) This will be important in estimating the decoherence of this set of histories.

Consider the history given by an unbroken string of N $\hat{\mathcal{P}}_0$ projections, corresponding to no photon being emitted during a time $N\delta t$.

The probability of such a history is given by the diagonal element of (16). We can expand the time evolution superoperator using (18) and see that after δt we get

$$\begin{aligned}\hat{\mathcal{P}}_0 e^{\mathcal{L}\,\delta t}(|\psi\rangle\langle\psi|\otimes|0\rangle\langle 0|)\hat{\mathcal{P}}_0 &\approx (e^{-i(\hat{H}_0 - i(\kappa^2/G)\hat{a}^\dagger\hat{a})\delta t}|\psi\rangle\langle\psi| \\ &\quad \times e^{i(\hat{H}_0 + i(\kappa^2/G)\hat{a}^\dagger\hat{a})\delta t})|0\rangle\langle 0|.\end{aligned} \tag{20}$$

Repeating this N times, and taking the trace, we get

$$p(h) \approx \mathrm{Tr}\{e^{-\hat{H}_{\mathrm{eff}}N\delta t}|\psi\rangle\langle\psi|e^{i\hat{H}_{\mathrm{eff}}^\dagger N\delta t}\}, \tag{21}$$

which agrees exactly with the probability of the quantum jump trajectory when no jumps are detected.

Suppose now that at time $N\delta t$ a photon is emitted, so that instead of using a final projection $\hat{\mathcal{P}}_0$ we use $\hat{\mathcal{P}}_1$. This corresponds to keeping the ρ_{11} component of $\exp(\mathcal{L}\,\delta t)\rho$ instead of ρ_{00}, and yields a probability

$$p(h) \approx (2\delta t\kappa^2/G)\,\mathrm{Tr}\{\hat{a}e^{-i\hat{H}_{\mathrm{eff}}N\delta t}|\psi\rangle\langle\psi|e^{i\hat{H}_{\mathrm{eff}}^\dagger N\delta t}\hat{a}^\dagger\}, \tag{22}$$

Once again, this agrees exactly with the probability of the corresponding quantum jump trajectory. What happens after the output mode has "registered" as being in the excited state? Essentially, there are two possibilities: Either the output mode can drop back to the unexcited state (representing absorption of the photon by the measuring device) or it can remain in the excited state.

$$\hat{\mathcal{P}}_0 e^{\mathcal{L}\,\delta t}(|\psi'\rangle\langle\psi'| \otimes |1\rangle\langle 1|)\hat{\mathcal{P}}_0 \approx \Gamma_1\delta t|\psi'\rangle\langle\psi'| \otimes |0\rangle\langle 0|\,, \quad (23)$$

$$\hat{\mathcal{P}}_1 e^{\mathcal{L}\,\delta t}(|\psi'\rangle\langle\psi'| \otimes |1\rangle\langle 1|)\hat{\mathcal{P}}_1 \approx (1 - \Gamma_1\delta t)e^{-i\hat{H}_{\rm eff}\delta t}|\psi'\rangle\langle\psi'|e^{i\hat{H}_{\rm eff}^{\dagger}\delta t} \otimes |1\rangle\langle 1|\,. \quad (24)$$

We see that the output mode has a probability of roughly $\Gamma_1\delta t$ per time δt of dropping back to the ground state, while the system state continues to evolve according to the effective Hamiltonian $\hat{H}_{\rm eff}$.

This is slightly different from quantum jumps. Quantum jumps are resolved only on a time scale $1/\Gamma_1$, not $\delta t \ll 1/\Gamma_1$. However, there is a near-unity probability of the external mode returning to the ground state within a time of order $1/\Gamma_1$, so one can simply sum over all histories in which the photon is absorbed within this time. It is easy to see that these will, once again, match the quantum jump trajectories exactly. This type of coarse graining is common in decoherent histories [10,11], and does not alter the form of the result.

By combining the three cases described in this section, one can produce histories of multiple jumps. It is clear that the probability of such a history will be exactly of the form (12).

In order for this discussion of probabilities to be meaningful, we must require the histories to be decoherent. Exact decoherence is a very difficult criterion to meet. It is more usual to show that a model is *approximately* decoherent. In order for the probability sum rules to be satisfied to a precision $\epsilon \ll 1$, we require that [20]

$$|D[h,h']|^2 < \epsilon^2 D[h,h]D[h',h'] = \epsilon^2 p(h)p(h') \quad (25)$$

for all distinct histories h, h'. Generally speaking, the more "different" a pair of histories (i.e., the more projections they differ in), the more suppressed the off-diagonal term. So it suffices to look at two histories which differ at a single time t_i; one having a projection $\hat{\mathcal{P}}_0$ and the other $\hat{\mathcal{P}}_1$. This is equivalent to picking out the ρ_{01} or ρ_{10} component of $\exp(\mathcal{L}\,\delta t|\psi'\rangle\langle\psi'|$ at that time.

Examining the components given by Eqs. (18) and (19),

$$\frac{|D[h,h']|^2}{p(h)p(h')} \sim \frac{1}{(G\delta t)^2}\,, \quad (26)$$

we expect the sum rules to be obeyed with a precision of roughly $O(1/G\delta t)$ (where we once again have assumed $\kappa\langle\hat{a}^\dagger\hat{a}\rangle$ is small compared with Γ_1).

We have seen how, in this simple model of a continuous measurement, the set of quantum jump trajectories corresponds to a set of decoherent histories. One of the principal goals of the decoherent histories program was to create a formalism which would reproduce the results of the usual Copenhagen formalism in measurement situations. It is pleasant to note that extensions to repeated or continuous measurements follow naturally within decoherent histories.

In this Letter, I considered only one measurement scheme, direct photodetection. In fact, there are many different schemes which give rise to different unravelings of the same master equation—heterodyne and homodyne detection, to name two [4,21]. I have no doubt that arguments similar to those I have advanced in this paper will demonstrate similar correspondences to different sets of decoherent histories.

This correspondence also has obvious practical benefits. Enumerating a full set of decoherent histories and calculating their probabilities is an arduous and unrewarding task, in general. There is a great deal of accumulated experience in simulating quantum trajectories; in situations where one would like to generate individual decoherent histories with correct probabilities, existing numerical techniques could be used.

The decoherent histories formalism was developed largely in response to the problems of quantum cosmology, while quantum trajectories arose from problems in quantum optics and atomic physics. Both extend the von Neumann description of quantum mechanics to new realms of application. As the connections between the two formalisms are further explored, we can hope that a great deal of interesting physics will emerge.

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Note added.—After completion of this research, I became aware of related work by Ting Yu from a rather different approach [22].

*Current address: Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030. Electronic address: brun@itp.ucsb.edu

[1] H. J. Carmichael, S. Singh, R. Vyas, and P. R. Rice, Phys. Rev. A **39**, 1200 (1989).
[2] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
[3] C. W. Gardiner, A. S. Parkins, and P. Zoller, Phys. Rev. A **46**, 4363 (1992).
[4] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 1652 (1993).

[5] H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, Lecture Notes in Physics (Springer-Verlag, Berlin, 1993).
[6] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[7] R. Griffiths, J. Stat. Phys. **36**, 219 (1984).
[8] R. Omnès, J. Stat. Phys. **53**, 893, 933, 957 (1988).
[9] R. Omnès, J. Stat. Phys. **57**, 359 (1989).
[10] M. Gell-Mann and J. B. Hartle, in *Complexity, Entropy, and the Physics of Information*, edited by W. Zurek, Santa Fe Institute Studies in the Science of Complexity (Addison-Wesley, Reading, MA, 1990), Vol. VIII.
[11] M. Gell-Mann and J. B. Hartle, Phys. Rev. D **47**, 3345 (1993).
[12] R. Omnès, *The Interpretation of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 1994).
[13] L. Diósi, N. Gisin, J. J. Halliwell, and I. C. Percival, Phys. Rev. Lett. **74**, 203 (1995).
[14] J. P. Paz and W. Zurek, Phys. Rev. D **48**, 2728 (1993).
[15] L. Diósi, Phys. Lett. A **185**, 5 (1994).
[16] M. A. Nielsen, Quantum Semiclass. Opt. **8**, 237 (1996).
[17] W. Zurek, Phys. Rev. D **24**, 1516 (1981); W. Zurek, in *Frontiers of Nonequilibrium Statistical Physics*, edited by G. T. Moore and M. O. Scully (Plenum, New York, 1983).
[18] H. M. Wiseman and G. J. Milburn, Phys. Rev. A **47**, 642 (1993).
[19] M. Lax, Phys. Rev. **157**, 213 (1967).
[20] H. Dowker and J. Halliwell, Phys. Rev. D **46**, 1580 (1992).
[21] L. Diósi, in *Waves and Particles in Light and Matter*, edited by A. Garuccio and A. van der Merwe (Plenum, New York, 1994).
[22] T. Yu, Imperial College Report No. TP/95-96/44, gr-qc/9605071, 1996 (to be published).