Chaotic Hysteresis in an Adiabatically Oscillating Double Well

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We consider the motion of a damped particle in a potential oscillating slowly between a simple and a double well. The system displays hysteresis effects which can be of a periodic or chaotic type. We explain this behavior by computing an analytic expression of a Poincaré map. [S0031-9007(97)02442-3]

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Although hysteresis is a quite familiar and ubiquitous phenomenon, no general theory can actually describe its many facets. In condensed matter, hysteresis often accompanies a phase transition, which by nature results from the cooperative effect of a large number of degrees of freedom. This has recently led some authors to analyze it by means of Langevin type [1] or master equations [2] with infinitely many degrees of freedom, and to propose various scaling laws for the area of the hysteresis loop. A mean field treatment of this problem [3] reduces it to an ordinary differential equation for the order parameter, with some slowly time dependent external parameter such as the magnetic field. Noise can be incorporated into the problem. Similar equations appear naturally to describe mechanical or electrical systems, as well as lasers [4]. Hysteresis effects may appear if the equilibria of the dynamical system with a static parameter undergo a bifurcation, and scaling laws have also been found in this case [5].

A paradigmatic example is the motion of a damped particle in a slowly varying potential. Periodically forced and damped nonlinear oscillators such as the Duffing [6] and Van der Pol [6,7] oscillator have been extensively studied. One usually chooses to consider either a linear external driving, or a modulated potential amplitude (nonlinear Mathieu equations). Here we consider a rather different type of forcing, which has, to our knowledge, not yet been analyzed, although it is physically quite natural as we shall show. Assume that the symmetric potential depends on a parameter smoothly interpolating between a simple and a double well, so that the static bifurcation diagram looks like the inset of Fig. 2. Imagine now that the parameter is oscillating slowly between these extreme values. The particle starting close to the initially stable origin does not react immediately to the bifurcation. Rather, it remains for some time close to the now unstable origin, before falling into one of the newly formed minima, which it follows until it merges again with the origin. Thus this *bifurcation delay*, which has been observed in various physical systems [4] and was rigorously analyzed for the first time by Neishtadt [8], is responsible for metastability leading to hysteresis.

In this Letter we address the following question: For a periodic parameter variation, which symmetric potential well will the particle fall into during each cycle? In the overdamped case, it always chooses the same minimum [Fig. 1(a)]. But, at low friction, we found that, depending on the frequency of the parameter variation, the particle may also fall alternatively into the left and right equilibrium [Fig. 1(b)] or even go from one minimum to the other in a random way [Fig. 1(c)]. We observed the same phenomenon in an experimental realization of the system, a pendulum on a rotating table. In this work, we show that this surprising behavior can be understood by means of a Poincaré map that we compute explicitly to lowest order of the parameter variation. We only outline the derivation of this fairly complicated, although essentially one-dimensional map, postponing rigorous proofs for future publication [9].

The equation of motion of a damped particle in a slowly varying potential can be written as

$$\ddot{q} + 2\gamma\dot{q} + \Phi'(q, \lambda(\varepsilon t)) = 0, \qquad (1)$$

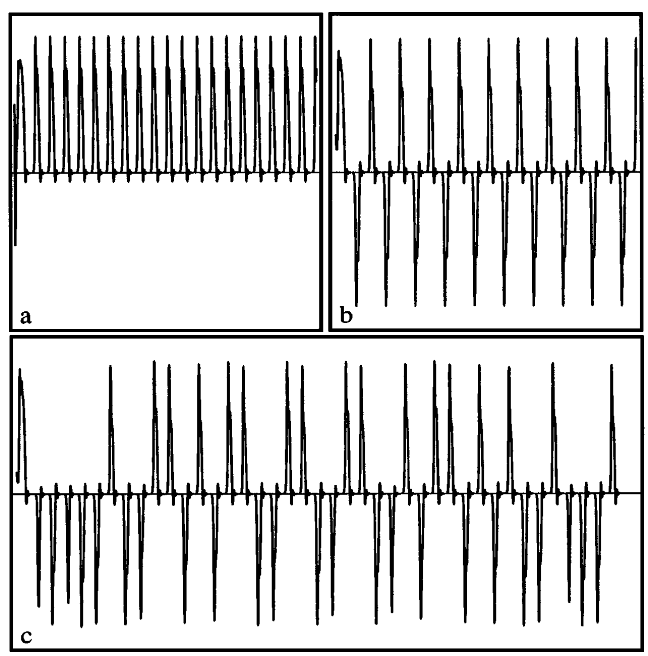

FIG. 1. Position of the particle (or the pendulum) as a function of time, each peak corresponding to the particle falling into one of the wells. The sequence of visited wells may be periodic, biperiodic, or chaotic.

where the prime denotes derivation with respect to q and $0 < \varepsilon \ll 1$ is the adiabatic parameter.

Introducing the variables $x = (x_1, x_2) = (q, \dot{q})$, this equation can be transformed into the nonautonomous first order system

$$\begin{aligned} \varepsilon \dot{x}_1 &= x_2 , \\ \varepsilon \dot{x}_2 &= -2\gamma x_2 - \Phi'(x_1, \lambda(\tau)) , \end{aligned} \tag{2}$$

where the dots now indicate derivation with respect to the slow time $\tau = \varepsilon t$.

For fixed $\lambda(\tau) \equiv \lambda_0$, the dynamics of (2) are well known. Around the equilibria $x_1 = q^*(\lambda_0)$, $x_2 = 0$, where $\Phi'(q^*(\lambda_0), \lambda_0) = 0$, the linearization of (2) has eigenvalues $a_\pm = -\gamma \pm \sqrt{\gamma^2 - \phi''}$, with $\phi'' = \Phi''(q^*(\lambda_0), \lambda_0)$. Hence the fixed point is a saddle if $\phi'' < 0$, a stable node if $0 < \phi'' < \gamma^2$, and a stable focus if $\phi'' > \gamma^2$. With this information, the phase portrait is easily drawn.

We will consider potentials of the following type: $\Phi(q, \lambda)$ is an even analytic function of q such that the origin O is hyperbolic when $\lambda > 0$, a node for $\lambda_-(\gamma) < \lambda < 0$, and a focus for $\lambda < \lambda_-$. For positive λ, Φ has two minima $\pm q^*(\lambda)$ which are nodes when $\lambda < \lambda_+(\gamma)$ and focuses when $\lambda > \lambda_+$.

The simplest example is the *Ginzburg-Landau potential* $\Phi(q, \lambda) = -\frac{1}{2}\lambda q^2 + \frac{1}{4}q^4$ (here $\lambda_- = -\gamma^2$, $\lambda_+ = \gamma^2/2$), q being the order parameter and λ the difference between the temperature and its critical value. But there is also a simple mechanical system which can be described by an equation of this type, namely, a plane pendulum on a rotating table. In the frame rotating with frequency Ω, it experiences a torque $-LMg\sin q$ due to its weight and a centrifugal torque $I\Omega^2 \sin q \cos q$, where L is the distance between suspension point P and center of mass G, I the moment of inertia with respect to P, and q the angle between PG and the vertical. Taking as a time unit $\Omega_{\text{cr}}^{-1} = (LMg/I)^{1/2}$, we obtain the equation of motion (1) with $\Phi'(q, \lambda) = \sin q[1 - (\lambda + 1)\cos q]$, $\lambda = \Omega^2 - 1$.

Finally, the function $\lambda(\tau)$ we consider is periodic with period 1, and has exactly one minimum and one maximum. In order to analyze the behavior of the dynamical system, we want to compute the Poincaré map in the (x_1, x_2) plane during one period to dominant order in ε. This proceeds essentially by following the motion of x along its static equilibria. Let us denote by $a_\pm^o(\tau)$ and $a_\pm^*(\tau)$ the eigenvalues of the linearization around O and $q^*(\lambda(\tau))$, respectively, and use the notation

$$\begin{aligned} \alpha^{o,*}(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \mathrm{Re}[a_+^{o,*}(\tau)]\,d\tau , \\ \phi^{o,*}(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \mathrm{Im}[a_+^{o,*}(\tau)]\,d\tau , \\ \delta^{o,*}(\tau_1, \tau_2) &= \int_{\tau_1}^{\tau_2} \mathrm{Re}[(a_+^{o,*} - a_-^{o,*})(\tau)]\,d\tau . \end{aligned}$$

We first study (2) in a neighborhood of the origin. The linearized system $\varepsilon \dot{x} = A(\tau)x$ can be solved by constructing a linear change of variables $x = S(\tau, \varepsilon)y$ transforming the equation into $\varepsilon \dot{y} = B(\tau, \varepsilon)y$, where B is as simple as possible. If $A(\tau)$ has distinct eigenvalues, one can find S regular in a neighborhood of $\varepsilon = 0$ such that B has exponentially small off-diagonal terms. Complete diagonalization is possible if we require S to admit only an asymptotic expansion in ε [10]. The full system (2) thus becomes

$$\begin{aligned} \varepsilon \dot{y}_1 &= a_+^o(\tau, \varepsilon) y_1 + b_+(y_1, y_2; \tau, \varepsilon) , \\ \varepsilon \dot{y}_2 &= a_-^o(\tau, \varepsilon) y_2 + b_-(y_1, y_2; \tau, \varepsilon) , \end{aligned} \tag{3}$$

with $a_\pm^o(\tau, \varepsilon) = a_\pm^o(\tau) + \mathcal{O}(\varepsilon)$ and $b_\pm = \mathcal{O}(|y|^3)$. We assume that $a_+^o(\tau_0) > 0$ and define an *adiabatic unstable manifold* $y_2 = u(y_1; \tau, \varepsilon)$ by the equation

$$\varepsilon \dot{u} = a_-^o u + b_-(y_1, u) - \partial_y u[a_+^o y_1 + b_+(y_1, u)] ,$$

where the initial condition coincides with the instantaneous unstable manifold at $\tau = \tau_0$. The solution, whose asymptotic expansion can be computed, may be continued to all times such that the origin is a saddle or a node. We then carry out the change of variables $y_2 = u(y_1) + \eta$ and define in a similar way an *adiabatic stable manifold* $y_1 = v(\eta; \tau, \varepsilon)$. With $y_1 = v(\eta) + \xi$, Eq. (3) becomes

$$\begin{aligned} \varepsilon \dot{\xi} &= [a_+^o(\tau, \varepsilon) + \beta_+(\xi, \eta; \tau, \varepsilon)]\xi , \\ \varepsilon \dot{\eta} &= [a_-^o(\tau, \varepsilon) + \beta_-(\xi, \eta; \tau, \varepsilon)]\eta , \end{aligned} \tag{4}$$

with $\beta_\pm = \mathcal{O}(\xi^2 + \eta^2)$.

In the overdamped case, when $\lambda(\tau)$ always remains in the interval (λ_-, λ_+), the problem is easy to analyze because it can be reduced to a one-dimensional one. Assume that $\lambda(\tau) > 0$ for $0 < \tau < \tau_b$ and $\lambda(\tau) < 0$ for $\tau_b < \tau < 1$. One shows that it is always possible to transform (2) into (4), with $a_-^o + \beta_- < 0$. Thus, we conclude from the second equation that $\eta(\tau)$ will go to zero exponentially fast, and it is sufficient to consider the reduced equation on the adiabatic manifold, $\varepsilon \dot{\xi} = [a_+^o + \beta_+(\xi, 0)]\xi \equiv g(\xi; \tau, \varepsilon)$, which undergoes a direct pitchfork bifurcation at $\tau = \mathcal{O}(\varepsilon)$, and an inverse pitchfork bifurcation at $\tau = \tau_b + \mathcal{O}(\varepsilon)$.

The most important effect of the adiabaticity of the parameter variation is that the bifurcation is *delayed*. The orbit starting with a positive ξ at $\tau_0 \in (\tau_b - 1, 0)$ reaches the $\mathcal{O}(\varepsilon)$ neighborhood of the origin at $\tau_1 = \tau_0 + \mathcal{O}(\varepsilon|\ln \varepsilon|)$. As long as $\xi = \mathcal{O}(\varepsilon)$, we have $\varepsilon \dot{\xi} = [a_+^o(\tau) + \mathcal{O}(\varepsilon)]\xi$ so that

$$\xi(\tau) = \xi(\tau_1) \exp\left(\frac{1}{\varepsilon}[\alpha^o(\tau_1, \tau) + \mathcal{O}(\varepsilon)]\right).$$

The smallness of ξ is thus guaranteed as long as $\alpha^o < 0$. For $\tau_1 < \tau < 0$, α^o is decreasing; it has increased again to zero only at $\tau = \Psi(\tau_1) > 0$, which is by definition the *delayed bifurcation time*. Afterwards, the orbit quickly jumps on the upper branch q^* and follows it adiabatically until the inverse bifurcation at τ_b, where we can prove that $\xi(\tau_b) = C\varepsilon^{1/4}$. The next delayed bifurcation occurs at $\hat{\tau} = \Psi(\tau_b)$.

It follows that the Poincaré map, defined by $\xi(\hat\tau + 1) = T(\xi(\hat\tau))$, is a monotonic odd function of ξ, such that for $\xi > \exp[-\frac{1}{\varepsilon}\alpha^o(\hat\tau, \hat\tau + 1)]$ one has $0 < c_1\varepsilon^{1/4} < T(\xi) < c_2\varepsilon^{1/4}$. This implies that the map has three fixed points, the unstable origin and two symmetric stable points. The positive solution corresponds to the attractive periodic orbit of Fig. 1(a), and, if plotted in the (λ, ξ) plane, to an asymptotic hysteresis cycle (see inset of Fig. 2), whose area we are able to prove scales the same as $\mathcal{A}(\varepsilon) \simeq \mathcal{A}(0) + c\varepsilon^{3/4}$.

We now turn to the most complicated case, when the amplitude of $\lambda(\tau)$ is large enough for the origin to be a focus for $\tau_-^o < \tau < \tau_+^o$ and for q^* to be a focus for $\tau_-^* < \tau < \tau_+^*$. We make a Poincaré section at the delay time $\hat\tau = \Psi(\tau_b - 1)$, which we assume to be in the interval (τ_-^*, τ_+^*) (Fig. 2).

The variables $\zeta = (\xi, \eta)$ are defined for $\tau \notin [\tau_-^o, \tau_+^o]$ and $|\xi| < d$, where d is a constant such that x_1 is closer to the origin than the focus at q^*. We want to compute $\zeta(\hat\tau + 1) = (\xi_1, \eta_1)$ as a function of $\zeta(\hat\tau) = (\xi_0, \eta_0)$ in the case where $\xi_0 \in (0, d)$ and $\eta_0 = \mathcal{O}(\varepsilon)$. From (4), we deduce that $\xi(\tau) = d$ at $\tau = \bar\tau(\xi_0) + \mathcal{O}(\varepsilon)$, where $\bar\tau(\xi_0)$ is the solution of

$$\alpha^o(\hat\tau, \bar\tau) = -\varepsilon \ln(\xi_0/d). \qquad (5)$$

For most trajectories, $\bar\tau$ is very close to $\hat\tau$, but orbits which are exponentially close to the origin at $\hat\tau$ can be appreciably delayed. $\bar\tau(\xi_0)$ is a decreasing function of ξ_0 for $\xi_c \le \xi_0 \le d$, with $\bar\tau(d) = \hat\tau$ and $\bar\tau(\xi_c) = \tau_+^*$, where $\xi_c = d\exp[-\frac{1}{\varepsilon}\alpha^o(\hat\tau, \tau_+^*)]$ [for practical purposes, one may approximate $\bar\tau(\xi_0)$ by $\hat\tau - \varepsilon c \ln(\xi_0/d)$].

Next, we construct a particular solution of (2), $\bar x(\tau)$, which remains in the neighborhood of the upper branch $q^*(\tau)$ and such that $\bar\xi(\tau_b) = C\varepsilon^{1/4}$, $\bar\eta(\tau_b) = 0$. Writing $x = \bar x(\tau) + y$, we obtain the equation $\varepsilon\dot y = \bar A(\tau; \varepsilon)y + b(y, \tau)$, where $\bar A$ has eigenvalues $a_\pm^*(\tau) + \mathcal{O}(\varepsilon^{1/2})$. The nonlinear term $b(y, \tau)$ may be decreased to order ε by the change of variables $y = z + \chi(z)$, which puts the instantaneous system into normal form. When $\tau \neq \tau_+^*$, the linear part of the equation can be diagonalized. Then we have to take care of the turning point τ_+^*, where the eigenvalues of $\bar A(\tau)$ cross and $S(\tau)$ diverges. The problem is solved by carrying out, in the interval $[\tau_+^* - \varepsilon^{2/3}, \tau_+^* + \varepsilon^{2/3}]$, a change of variables which puts B into Jordan form, so that one obtains Airy's equation. Putting together these steps, we finally obtain

$$\begin{aligned} \xi(\tau_b) &= \bar\xi(\tau_b) + e^{\alpha_*/\varepsilon}\sin(\phi_*/\varepsilon), \\ \eta(\tau_b) &= e^{(\alpha_* - \delta^*)/\varepsilon}\sin(\phi_*/\varepsilon + \theta^*), \end{aligned} \qquad (6)$$

where $\delta^* = \delta^*(\tau_+^*, 1) + \mathcal{O}(\varepsilon^{1/2})$ is a constant and $\alpha_* = \alpha_*(\xi_0) + \mathcal{O}(\varepsilon^{1/2})$, $\phi_* = \phi_*(\xi_0) + \mathcal{O}(\varepsilon)$, with

$$\begin{aligned} \alpha_*(\xi_0) &= \alpha^*(\bar\tau(\xi_0), 1), \\ \phi_*(\xi_0) &= \phi^*(\bar\tau(\xi_0), \tau_+^*). \end{aligned} \qquad (7)$$

Of course $\zeta(\tau_b)$ also depends on η_0 via $y(\bar\tau)$, but only at next-to-leading order. For $\xi_0 \in [\xi_c, d]$, Eq. (6) is the parametric equation of an exponentially squeezed spiral.

Proceeding in a similar way around the origin, we find $\zeta(\hat\tau + 1) = U\zeta(\tau_b)$, with

$$U = \begin{pmatrix} \sin\left(\frac{\phi^o}{\varepsilon}\right) & e^{-\delta_2^o/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon} + \theta_2^o\right) \\ e^{-\delta_1^o/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon} + \theta_1^o\right) & e^{-\delta_3^o/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon} + \theta_3^o\right) \end{pmatrix}, \qquad (8)$$

where $\phi^o = \phi^o(\tau_-^o, \tau_+^o) + \mathcal{O}(\varepsilon^{1/2})$ is the dynamic phase, $\delta_1^o = \delta^o(0, \tau_-^o) + \mathcal{O}(\varepsilon^{1/2})$, $\delta_2^o = \delta^o(\tau_+^o, \hat\tau + 1) + \mathcal{O}(\varepsilon^{1/2})$, and $\delta_3^o = \delta_1^o + \delta_2^o$. The geometric phase shifts θ_i can be expressed in terms of Airy functions (plus corrections of order $\varepsilon^{1/3}$). This transformation acts like a rotation of angle ϕ^o/ε and an exponential contraction along the stable manifold.

Combining (6) and (8), we finally obtain the expression of the Poincaré map $\xi_1 = T_1(\xi_0, \eta_0; \varepsilon)$, $\eta_1 = T_2(\xi_0, \eta_0; \varepsilon)$ with

$$\begin{aligned} T_1 = {} & \sin\left(\frac{\phi^o}{\varepsilon}\right)\left[C\varepsilon^{1/4} + e^{\alpha_*/\varepsilon}\sin\left(\frac{\phi_*}{\varepsilon}\right)\right] \\ & + e^{(\alpha_* - \delta^* - \delta_2^o)/\varepsilon}\sin\left(\frac{\phi^o}{\varepsilon} + \theta^0\right)\sin\left(\frac{\phi_*}{\varepsilon} + \theta^*\right) \end{aligned} \qquad (9)$$

and $T_2 = \mathcal{O}(e^{-\delta_1^o/\varepsilon})$.

The expression (9) of T_1 is valid for $\xi_0 > \xi_c$. By symmetry, T_1 is an odd function of ξ_0, and for $|\xi_0| < \xi_c$, it is monotonic as in the overdamped case. All variables appearing in (9) are independent of ξ_0, η_0 at lowest order in ε, except α_* and ϕ_* which are given by (7).

Since η is exponentially contracted at each iteration, we may replace the two-dimensional invertible Poincaré map by the noninvertible one-dimensional map $\xi_0 \mapsto T_1(\xi_0, 0)$. Its graph is oscillating around $C\varepsilon^{1/4}\sin(\phi^o/\varepsilon)$ with increasing amplitude and frequency as $\xi \searrow \xi_c$. One can prove that there is a positive constant μ such that when $\sin(\phi^o/\varepsilon) > e^{-\mu/\varepsilon}$, T_1 admits only one positive fixed point at $\xi \sim \varepsilon^{1/4}\sin(\phi^o/\varepsilon)$, which is stable. In this case, we obtain a hysteresis cycle of period 1. Similarly,

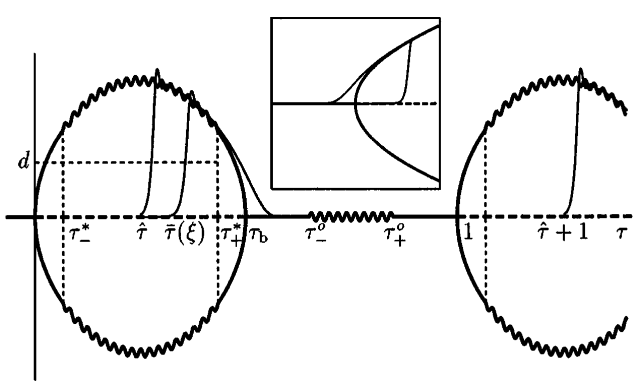

FIG. 2. Fixed points in the (τ, x_1) plane. Full lines represent nodes, dashed lines represent hyperbolic points, and wavy lines represent focuses. The thin lines are orbits of the system. The inset shows the static bifurcation diagram in the (λ, x_1) plane (thick lines) with an asymptotic hysteresis cycle (thin lines).

when $\sin(\phi^o/\varepsilon) < -e^{-\mu/\varepsilon}$, the map has a stable orbit of period 2, corresponding to the particle falling alternatively into the left and right well.

These predictions agree very well with numerical simulations of the rotating pendulum (Fig. 3). If $\sin(\phi^o/\varepsilon)$ is sufficiently positive, the pendulum performs an integer number of oscillations around the origin. After a few periods, it reaches an asymptotic hysteresis cycle with the same period as the forcing [Fig. 1(a)], characterized by $\xi(\hat{\tau} + n) \sim \varepsilon^{1/4}\sin(\phi^o/\varepsilon)$. When $\sin(\phi^o/\varepsilon)$ is sufficiently negative, the pendulum oscillates a half-integer number of times around the origin. Asymptotically, it follows a cycle with *twice* the driving period [Fig. 1(b)], described by $\xi(\hat{\tau} + n) \sim (-1)^n \varepsilon^{1/4}\sin(\phi^o/\varepsilon)$.

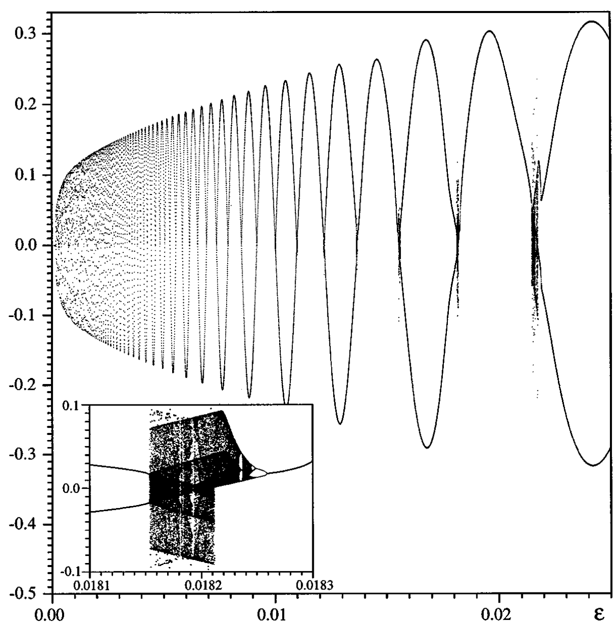

FIG. 3. Numerically computed bifurcation diagram of the Poincaré map for the rotating pendulum. For each value of ε on the abscissa, we plot the asymptotic behavior of $\xi(\hat{\tau} + n)$, $n \in \mathbb{N}$, for *one* initial condition. Regions with a period-1 and period-2 cycle are separated by small chaotic zones, the inset shows an enlargement of the second zone from the right.

The most interesting situation occurs for intermediate values of the dynamical phase, when $|\sin(\phi^o/\varepsilon)| < e^{-\mu/\varepsilon}$. In this case, we observed both numerically and experimentally a great variety of behaviors, including period doubling cascades and chaotic orbits [see inset of Fig. 3 and Fig. 1(c)]. These "chaotic" zones occur at integer values of $k = \phi^o/\pi\varepsilon$, and the width and height of the nth zone are proportional to $e^{-(\mu\pi/\phi^o)n}$. This prediction has been confirmed numerically.

The map (9) certainly deserves further study. But the most important fact to us is that it explains accurately the alternates of periodic, biperiodic, and chaotic hysteresis, in accordance with numerical simulations as well as with the laboratory experiment of the rotating pendulum.

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[1] M. Rao, H.K. Krishnamurthy, and R. Pandit, Phys. Rev. B **42**, 856 (1990).
[2] W.S. Lo and R.A. Pelcovits, Phys. Rev. A **42**, 7471 (1990).
[3] T. Tomé and M.J. de Oliveira, Phys. Rev. A **41**, 4251 (1990).
[4] P. Mandel and T. Erneux, Phys. Rev. Lett. **53**, 1818 (1984); J. Stat. Phys. **48**, 1059 (1987).
[5] P. Jung *et al.*, Phys. Rev. Lett. **65**, 1873 (1990); A. Hohl *et al.*, Phys. Rev. Lett. **74**, 2220 (1995).
[6] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer-Verlag, New York, 1983).
[7] E.F. Mishchenko, Yu.S. Kolesov, A.Yu. Kolesov, and N.Kh. Rozov, *Asymptotic Methods in Singularly Perturbed Systems* (Consultants Bureau, New York, 1994).
[8] A.I. Neishtadt, Diff. Eq. **23**, 1385 (1987); Diff. Eq. **24**, 171 (1988).
[9] N. Berglund and H. Kunz (to be published).
[10] W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations* (Krieger, New York, 1976).