Strange Waves in Coupled-Oscillator Arrays: Mapping Approach

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A phase model of rings of coupled oscillators is proposed and shown to exhibit a peculiar type of wave. As a parameter is varied, such waves are born with a characteristic pattern and then develop into complex waves such that phase differences between neighboring oscillators are *spatially* "chaotic," showing type-3 intermittency. Their behavior is studied on the basis of a multivalued one-dimensional map obeyed by the phase differences. [S0031-9007(97)02511-8]

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Large populations of coupled limit-cycle oscillators have been actively studied in recent years (see [1–3], and references therein). Such dynamical systems are useful as models for a variety of far from equilibrium systems, for example, diverse physiological organs such as gastrointestinal tracts [1], convecting fluids, and arrays of Josephson junctions (e.g., [4,5]). Besides these "classic" examples, the recent discovery of 40 Hz oscillations in a mammalian visual cortex has renewed interest in coupled limit-cycle oscillators in a neuroscientific context [6]. Assemblies of interacting limit-cycle oscillators may be viewed as an important category of large-scale dynamical systems which are now in vogue in nonlinear dynamics. Recent studies have demonstrated that such assemblies possess a rich spectrum of interesting behavior comparable to those of other types of systems. An important example is macroscopic synchronization in which a macroscopic number of element oscillators are mutually entrained with a common frequency, and the onset of which has some unique features in comparison with conventional phase transitions [2,3].

Most of the previous studies, however, have been carried out for globally coupled systems in which each element is linked to every other in an identical way. Global coupling is a convenient choice for theoretical studies, having made a variety of investigations possible. By contrast, the behavior of populations of *locally* coupled oscillators has not yet been extensively studied. What is now important would be to uncover the whole range of phenomena such systems can exhibit. This Letter addresses this issue for one-dimensional arrays of limit-cycle oscillators with nearest-neighbor interactions. Coupling as well as incoherency among native oscillations is assumed weak; hence models of the following form are available [2]: for $j = 1, \ldots, N(\gg 1)$,

$$d\theta_j/dt = \Omega_j + \epsilon h(\theta_{j-1} - \theta_j) + \epsilon h(\theta_{j+1} - \theta_j)\,, \tag{1}$$

where θ_j is the phase of oscillator j, Ω_j its natural frequency, ϵ the coupling strength, and $h(\theta) = h(\theta + 2\pi)$ the coupling function. Although there are some earlier studies for this type of model [7–10], they are mostly concerned with such cases that $h(\theta)$ is composed of fundamental harmonic alone like $\sin\theta$. This may be reasonable if the oscillators are set close to their Hopf bifurcation points to display nearly sinusoidal oscillations, but may lose validity if otherwise. Moreover, some fatal roles of $h(\theta)$'s higher harmonic components have been discovered recently in synchronization phenomena under the global coupling [3]. In view of these, the coupling function is here chosen as

$$h(\theta) = \sin\theta + a\cos 2\theta \tag{2}$$

with a as the control parameter. This function is the simplest among "generic" ones which not only have a higher harmonic but also are free from the special symmetry $h(-\theta) = -h(\theta)$, and has been already proposed for the case of global coupling [11]. Another more general form of $h(\theta)$ has been confirmed to yield similar behavior as reported below, but the analysis is much easier for (2). For simplicity, the oscillators are assumed to be identical, thus a set of trivial transformations of θ and t simplifying Eq. (1) as

$$d\theta_j/dt = h(\theta_{j-1} - \theta_j) + h(\theta_{j+1} - \theta_j)\,, \tag{3}$$

which form of equations will be used hereafter instead of (1). This Letter deals only with the case of periodic boundary conditions: $\theta_0 = \theta_N$, $\theta_{N+1} = \theta_1$. Numerical simulations have been performed for $N = 100$ by means of the fourth order Runge-Kutta-Gill method with time step of 0.03, under random initial conditions.

The main results of this work are as follows. It has been known so far that coupled-oscillator systems under similar settings exhibit traveling waves with constant phase differences. This Letter shows that in addition to them, the above system acquires a new novel type of solution beyond a critical value of a. Such solutions are temporally periodic but spatially "chaotic" in a way similar to chaos in one-dimensional maps. For a large, their spatial irregularity can be identified with type-3 intermittency. The new type of solution is thus important as a clear-cut example of spatial chaos. Another important point is that the behavior of such solutions can be explained

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from a multivalued map obeyed by successive phase differences. Such an approach should be useful, with proper extensions, for other phenomena and other types of locally coupled oscillators as well. Demonstration of its usefulness is another purpose of this Letter.

Let us begin by noting that there are two kinds of traveling wave solutions to (3) of the form $\theta_j = \Omega_e t + j\alpha$ with $\Omega_e = 2a\cos 2\alpha$: (1) $\alpha = \alpha_k = 2\pi k/N$, $(k = 0,\ldots,N-1)$; (2) $\alpha = \alpha_k = (2k+1)\pi/(N-2)$, $(k = 0,\ldots,N-3)$. The former with $\cos\alpha_k > 0$ are linearly stable, irrespective of a. None of the latter seem stable in the parameter range explored here, according to a numerical eigenvalue analysis. These solutions will hereafter be referred to as "simple waves." When a is increased from 0, only the simple waves of the first kind appear until a new type of solution begins to emerge for some of the initial conditions near $a = 2.28$. The new solutions are of the form $\theta_j = \Omega_e t + \psi_j$, where ψ_j are constants with a peculiar j dependence. As Fig. 1 exemplifies, near onset, such a solution consists of two simple-wave-like parts and a pair of transition regions. Among 100 initial conditions tested, most lead to simple waves for a near 2.28, but nearly half or more to the new kind of wave for a near 3 and larger. Its variety also grows with a: more or less different patterns of ψ_j appear depending on initial conditions. Figure 2 shows a typical wave pattern observed for $a = 6$, which is remarkably intricate while keeping the same basic feature as found near onset. The behavior of the phase differences between nearest elements, as is seen in Fig. 2(b), is especially curious, resembling an important category of trajectories in one-dimensional chaotic maps, so called intermittency [12]. Long laminar intervals plus occasional bursts define intermittency, which feature is indeed found in Fig. 2(b). The spatiotemporal behavior of the system is examined in Fig. 3; the nonsimple wave is roughly composed of two ordinary waves propagating in different directions for a close to 2.28, but for a large, there seems to be a mosaic of miniwaves with various sizes, as is understandable from the intermittency in Fig. 2(b).

To consider why all these happen, one may note that stationary values of $x_j \equiv \theta_{j+1} - \theta_j$ satisfy $\Omega_e = h(x_{j+1}) + h(-x_j)$. Then, using $\cos 2x = 1 - 2\sin^2 x$, it is easy to find

$$w_{j+1} = B \pm \sqrt{1 - w_j^2} \equiv F_\pm(w_j)\,, \tag{4}$$

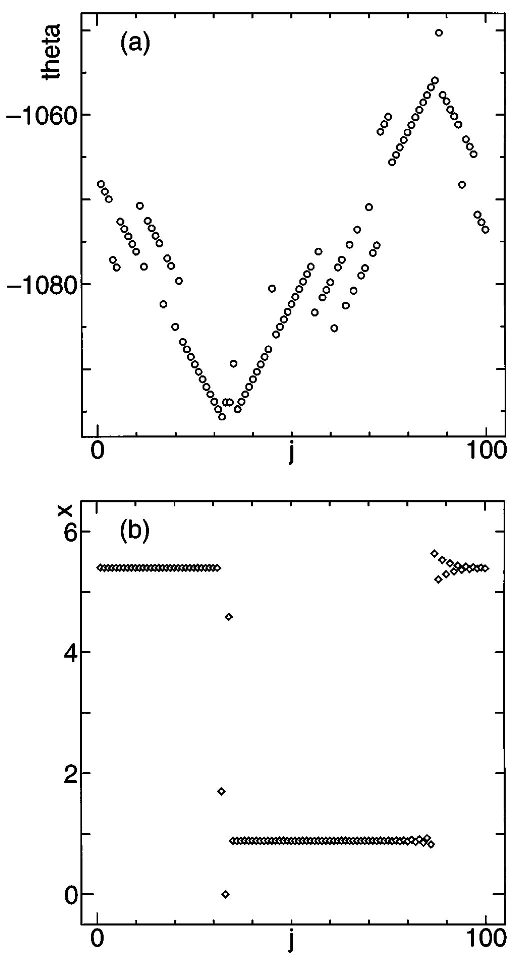

FIG. 1. Portraits of a nonsimple wave for $a = 2.3$ and $t = 1200$: (a) θ_j (raw value); (b) $x_j \equiv \theta_{j+1} - \theta_j$ (mod 2π).

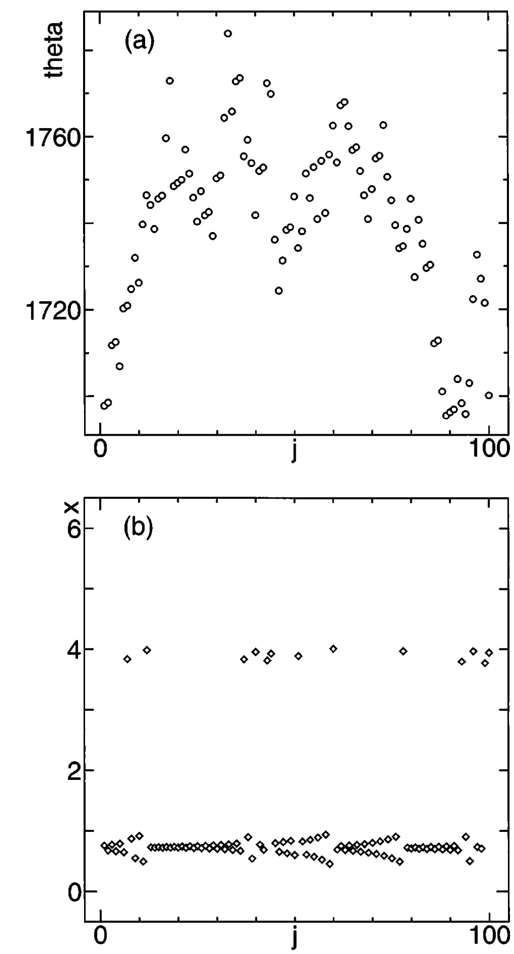

FIG. 2. Portrait of a nonsimple wave for $a = 6$ with the same details for (a) and (b) as in Fig. 1.

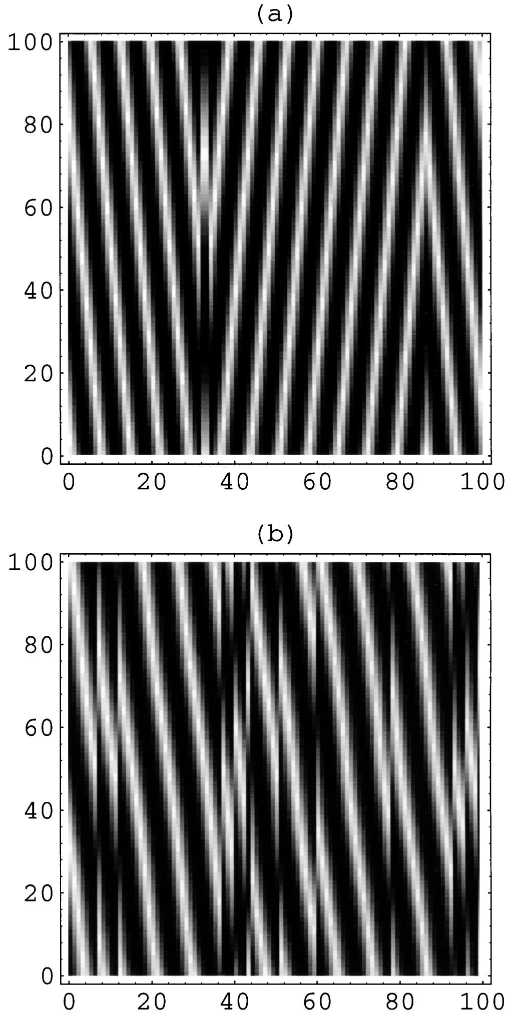

FIG. 3. The behavior of $\sin\theta_j$ on the (j, t) plane with lighter regions having larger values, where $\sin\theta$ is chosen just for illustration; (a) and (b) correspond to the waves of Figs. 1 and 2, respectively. The time axis (from bottom to top) has unit 0.09 in (a) and 0.06 in (b), covering nearly 1.3 times the period of the wave in (a) and 1.4 times in (b).

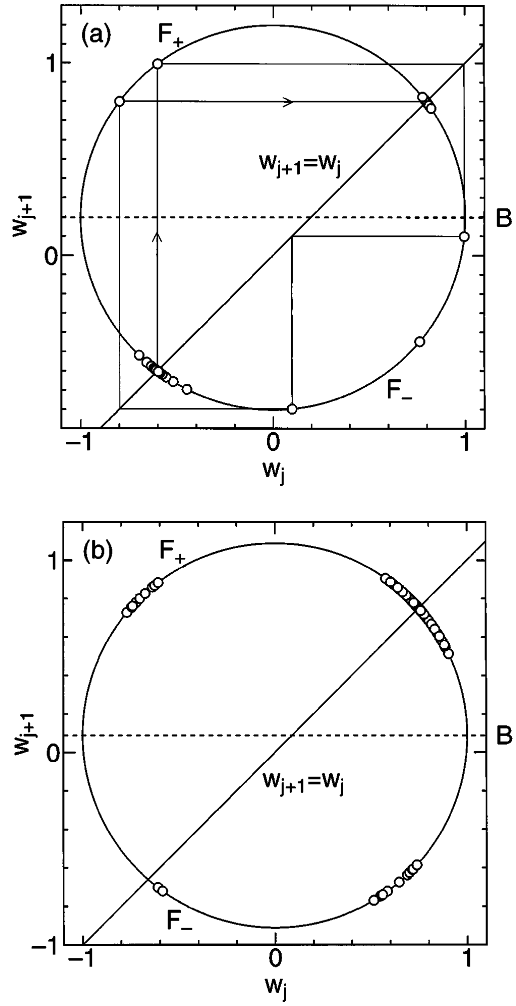

FIG. 4. The "circle map", Eq. (4): (a) a translation of Fig. 1(b) with the $m = 3$ heteroclinic route indicated by thin lines; (b) the same as in Fig. 2 except that t is here 3600. The value of B is 0.1966 in (a) and 0.088 77 in (b).

where $w_j \equiv B(1 + 4a\sin x_j)/2$ with $B \equiv (\frac{1}{2} - 2a\Omega_e + 4a^2)^{-1/2}$. This is a one-dimensional map composed of two branches corresponding to the upper and lower half of a unit circle (see Fig. 4). Suggested by the typical form of nonsimple waves near onset, let us assume that x_j remains constant with value α in a range $j \leq j_0$; B is then given by $(\frac{1}{2} + 8a^2\sin^2\alpha)^{-1/2}$, where α, hence Ω_e as well, depends on initial conditions, and in general differs from α_k for the simple waves. For $B < 1$, the minus branch of the map has a stable fixed point, $w = w_-$, which corresponds to $x = \alpha$, while the plus branch has an unstable fixed point, $w = w_+$, which corresponds to $x = 2\pi - \alpha$. In Fig. 1(b), the left (right) plateau is a manifestation of the stable (unstable) fixed point. Such a solution requires that w_j staying at w_- can change branches from the minus to the plus. A couple of necessary conditions for this are given by (i) $F_+(w_-) \leq 1$ and (ii) $F_+(w_-) \leq B(1 + 4a)/2$, as easily known from (4) and the definition of w_j, respectively, which lead to

$$a \geq \frac{9}{4} \equiv a_c\,, \tag{5}$$

$$\frac{1}{2a} - 1 \leq \sin\alpha \leq -\frac{7}{4a}\,. \tag{6}$$

Let us then consider how many intermediate steps are taken for w_j to reach the unstable fixed point, in other words, the size of the "ramp" [the left transition region in Fig. 1(b)], m, defined by $x_{j_0} = \alpha$ and $x_{j_0+m+1} = 2\pi - \alpha$. One can prove that $m = 0, 1, 2$ are all impossible and that when (6) is taken into account, a ramp of $m = 3$ may exist with $\sin\alpha = -(3 + \sqrt{17})/4a$ in the range

$$a \geq \frac{5 + \sqrt{17}}{4} = 2.280\,77\ldots \equiv a_3\,. \tag{7}$$

The nonsimple waves near onset were found to have $m = 3$ [see Fig. 4(a)]. Although nonsimple waves with

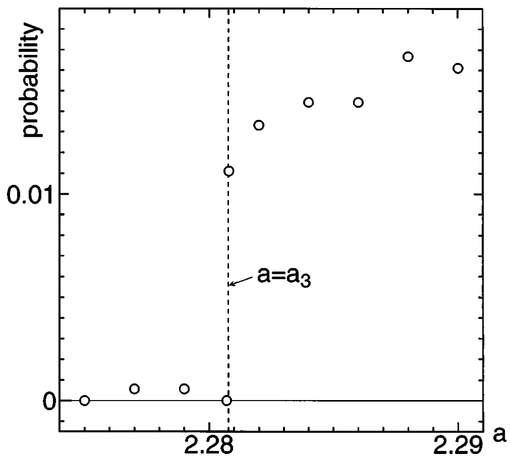

FIG. 5. The probability of hitting a nonsimple wave around $a = a_3$; 1800 samples of initial conditions were tested for each a.

m much larger than 3 were found within the interval $a_c < a < a_3$, the probability of coming across a nonsimple wave [13] abruptly drops virtually to zero as the parameter passes a_3 from above, as is evident in Fig. 5. Thus, practically, $a = a_3$ is the threshold for the appearance of the peculiar waves.

Note that the fixed point, $w = w_+$, is unstable with $dF_+/dw < -1$, thus x_j eventually departing from $2\pi - \alpha$ in an oscillatory way as j increases from $j_0 + m + 1$ until finally jumping back to the minus branch [see Fig. 1(b)]. As seen earlier, the whole wave is broken into pieces for a large, which fact may be roughly explained as follows: as its definition reveals, w_j can become large more and more easily as a grows, and hence tends to change branches more and more often because of the constraint $|w_j| \leq 1$. The "intermittency" as in Fig. 2(b) may be considered type 3 [12] because w_j is repeatedly reinjected around a fixed point which is oscillatory unstable. Interestingly, here, the multivaluedness of the map provides mechanisms of bursts and reinjections [cf. Fig. 4(b)]. Further increase in a results in more complex behavior as will be discussed elsewhere.

This Letter has shown for a model of rings of coupled oscillators that peculiar waves emerge when the anharmonicity of the coupling function exceeds a certain level and that their properties can be studied on the basis of a multivalued one-dimensional map for successive phase differences. The peculiarity of the waves for a large is due to the chaotic behavior of the phase differences. For convenience, one may call the new type of wave a *strange wave,* which is important as a clear-cut example of a spatially irregular but temporally regular phenomenon. As to spatially chaotic behavior, one may recall a quasisteady solution of the spatially continuous, complex Ginzburg-Landau equation [14,15], but it is chaotic temporally as well as spatially. The mapping approach proposed in this Letter would be applicable for a wide class of locally coupled oscillators. When each oscillator is given more than one degree of freedom, one may try to find a higher dimensional map of suitable local quantities. A different approach is developed in [9], which is available when the spatial variation of the system is fairly smooth. By contrast, the present approach would be particularly useful for phenomena characterized by wild spatial variations just as the strange waves.

The strong anharmonicity of $h(\theta)$ as treated here might appear, for example, in coupled highly relaxational oscillators where effectively strong nonlinearity excites higher harmonics. It should also be realizable in laboratory experiments (electronic circuits etc.) by devising such a coupling that Fourier components of a resulting $h(\theta)$ are controllable. However, the scenario presented above may not be the only one for the appearance of strange waves or similar phenomena. Hopefully this work triggers attempts at looking for them experimentally as well as numerically.

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