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Duality of x and ψ and a Statistical Interpretation of Space in Quantum Mechanics

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We introduce a "prepotential" $\mathcal{F}$ in quantum mechanics and show that the coordinate x is proportional to the Legendre transform of $\mathcal{F}$ with respect to the probability density. Inversion of the Schrödinger equation leads us to consider an x–ψ duality related to a modular symmetry. The scaling of x is determined by the "beta function," suggesting that in quantum mechanics the space coordinate is a macroscopic variable of a statistical system with $\hbar$ playing the role of scale. The formalism is extended to higher dimensions and to the Klein-Gordon equation. [S0031-9007(96)02145-X]

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In the past couple of years Seiberg-Witten theory [1] has shed new light on some aspects of supersymmetric quantum field theories. An important quantity in this theory is the prepotential $\mathcal{F}$ as it fixes the low-energy dynamics. In terms of $\mathcal{F}(\Phi)$, which is a holomorphic function of the chiral superfield Φ, one can express the dual variable $\Phi_D = \mathcal{F}'(\Phi)$ and the effective coupling constant $\tau = \mathcal{F}''(\Phi)$. The quantum moduli space of the theory is parametrized by the gauge invariant parameter $u = \langle \mathrm{Tr}\, \phi^2 \rangle$, where ϕ is the scalar component of Φ. In Seiberg-Witten theory a method has been developed to invert the function $a = a(u)$ to $u = u(a)$, where $a = \langle \phi \rangle$ [2]. In this theory a second-order differential equation is written down for the moduli parameters $a(u)$ and $a_D(u)$. The prepotential enables the inversion procedure and allows an interesting interpretation of second-order differential equations.

Following these ideas, we derive a method to invert the Schrödinger wave function $\psi = \psi(x)$ to $x = x(\psi)$. We define a "prepotential" $\mathcal{F}$ as function of ψ such that the dual variable $\psi_D = \partial \mathcal{F} / \partial \psi$ is a solution of the Schrödinger equation. In this formalism, the quantum dynamics is described by $\mathcal{F}$, which satisfies a nonlinear third-order differential equation which replaces the Schrödinger equation. The inversion formula shows that x is the Legendre transform of $\mathcal{F}$ with respect to the probability density, implying that in quantum mechanics the space may be seen as a macroscopic variable of a statistical system. In this context we show that the scaling properties of x with respect to $\tau = \partial_\psi^2 \mathcal{F}$ are determined in terms of the "beta function" $\hbar \partial_\hbar \tau$.

Let us consider the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m} \partial_x^2 + V(x) \right) \psi = E \psi \,, \qquad (1)$$

where E is in the physical spectrum of the Schrödinger operator. In a general Schrödinger problem, such as Eq. (1), for each E one can have one or two physical solutions. Let ψ_E denote a physical solution of Eq. (1) and ψ_{E_D} a solution of Eq. (1) linearly independent from ψ_E. We define the prepotential $\mathcal{F}_E$ by

$$\psi_{E_D} = \frac{\partial \mathcal{F}_E(\psi_E)}{\partial \psi_E} \qquad (2)$$

and consider

$$\partial_x \mathcal{F}_E = \psi_{E_D} \partial_x \psi_E = \frac{1}{2} \left[\partial_x (\psi_E \psi_{E_D}) + W \right], \qquad (3)$$

where by Eq. (1) the Wronskian $W = \psi_{E_D} \partial_x \psi_E - \psi_E \partial_x \psi_{E_D}$ is a constant. The crucial point is that Eq. (3) can be integrated exactly to

$$\mathcal{F}_E = \frac{1}{2} \psi_E \psi_{E_D} + \frac{W}{2} x + c \,, \qquad (4)$$

with c a constant which by Eq. (2) we can set to 0. It is easy to check that Eq. (4) is equivalent to

$$\mathcal{F}_E(\psi_E) = \psi_E^2\left[\frac{Wx_0 + \psi_{E0}\psi_{E_{D0}}}{2\psi_{E0}^2} - W\int_{\psi_{E_0}}^{\psi_E} dy\,\mathcal{G}_E(y)y^{-3}\right], \qquad (5)$$

where $\psi_{E0} \equiv \psi_E(x_0)$, $\psi_{E_{D0}} \equiv \psi_{E_D}(x_0)$, and the notation $x = \mathcal{G}_E(\psi_E)$ has been introduced in order to denote the functional dependence of x on ψ_E. By rescaling ψ_E we can set $W = -\frac{2\sqrt{2m}}{\hbar}$, so that we have

$$\frac{\sqrt{2m}}{\hbar}x(\psi_E) = \frac{1}{2}\psi_E\frac{\partial\mathcal{F}_E}{\partial\psi_E} - \mathcal{F}_E\,, \qquad (6)$$

which we rewrite in the "canonical form"

$$\frac{\sqrt{2m}}{\hbar}x(\psi_E) = \psi_E^2\frac{\partial\mathcal{F}_E}{\partial(\psi_E^2)} - \mathcal{F}_E\,, \qquad (7)$$

showing that the classical coordinate is proportional to the Legendre transform of the prepotential with respect to ψ_E^2. Duality of the Legendre transform yields

$$\frac{\hbar}{\sqrt{2m}}\mathcal{F}_E = \phi_E\partial_{\phi_E}x - x\,, \qquad (8)$$

where $\phi_E = \partial_{(\psi_E^2)}\mathcal{F}_E = \psi_{E_D}/2\psi_E$. Therefore $\mathcal{F}_E$ is the Legendre transform of $\frac{\sqrt{2m}}{\hbar}x$ and vice versa.

In a quantum mechanical problem we can distinguish two cases depending on whether ψ_E and $\overline{\psi}_E$ are or are not linearly dependent functions. In the former case (e.g., the harmonic oscillator) the quantity ψ_E^2 is nothing else but the (unnormalized) probability density ρ_E. Therefore, if $\overline{\psi}_E \propto \psi_E$, Eq. (7) implies that the classical coordinate x is proportional to the Legendre transform of the prepotential with respect to ρ_E. Furthermore, by Eqs. (6) and (8) it follows that

$$\rho_E \equiv \psi_E^2 = \frac{\sqrt{2m}}{\hbar}\partial_{\phi_E}x\,, \qquad (9)$$

that is, x is the generating function for the probability density at x itself.

Reality of the Schrödinger operator implies that $\overline{\psi}_E$ is still a solution of Eq. (1). Therefore, if $\overline{\psi}_E \not\propto \psi_E$ (e.g., in the case of the free particle, where $\psi_{E_D} = \overline{\psi}_E \propto \psi_E^{-1} \propto e^{ipx/\hbar}, E = p^2/2m$), then we can set

$$\psi_{E_D} = \overline{\psi}_E\,. \qquad (10)$$

[Note that with this choice W is purely imaginary. We can choose, without loss of generality, a normalization for ψ_E itself such that $W = -\frac{2i\sqrt{2m}}{\hbar}$.] Then by (7)

$$\rho_E \equiv |\psi_E|^2 = \frac{i\sqrt{2m}}{\hbar}x + 2\mathcal{F}_E\,, \qquad (11)$$

showing that the probability density of finding the particle at x is proportional to x itself with an additive correction which is proportional to the prepotential.

$\mathcal{F}$ plays a crucial role as it encodes the information on the microscopic theory. In particular, the Schrödinger equation can be replaced by $({}' \equiv \partial_{\psi_E})$

$$4\mathcal{F}_E''' + [V(x) - E)](\mathcal{F}_E' - \psi_E\mathcal{F}_E'')^3 = 0\,, \qquad (12)$$

where $\hbar$ appears only through $V(x)$ [with $x = x(\psi_E)$ given by (6)]. Equation (12) is obtained from Eqs. (1) and (6) by following the method introduced in [2,3]. In particular, by inverting the Schrödinger equation, we obtain

$$\frac{\hbar^2}{2m}\partial_{\psi_E}^2 x = \psi_E[E - V(x)](\partial_{\psi_E}x)^3, \qquad (13)$$

which can be seen as dual to Eq. (1). These dual formulations of quantum mechanics may generate different structures once one considers the second quantization or alternatively quantizes the expansion of x in powers of the wave function. In order to illustrate this point we first consider the dual power expansions

$$\psi_E = \sum_j \alpha_j^E x^j \Longleftrightarrow x = \sum_k \beta_k^E \psi_E^k \qquad (14)$$

and note that their structure suggests considering the x-ψ_E duality as reminiscent of the "mirror symmetry phenomenon" first observed for Calabi-Yau threefolds. We note that a similar remark was made in connection with the differential equation (and its inverse) satisfied by the generating function for Weil-Petersson volumes of moduli spaces of punctured Riemann spheres [4].

Before considering the quantization of (14), it is worth noticing that the above structures are related to the modular symmetry which underlies quantum mechanics. In particular, the relation among the space coordinate, the prepotential, and the wave function is related to the basic fact that any linear combination of ψ_E and ψ_{E_D} is still a solution of the Schrödinger equation. For the same reason, the formalism is invariant under the transformations

$$\tilde{\psi}_{E_D} = A\psi_E + B\psi_{E_D}, \qquad \tilde{\psi}_E = C\psi_E + D\psi_{E_D}\,, \qquad (15)$$

implying, in particular, that Eqs. (6) and (12) are modular invariant. This symmetry can also be explicitly checked by using the transformation properties of $\mathcal{F}_E$, which follow by comparing $\tilde{\psi}_{E_D} = \partial\widetilde{\mathcal{F}}(\tilde{\psi}_E)/\partial\tilde{\psi}_E$ with (15),

$$\begin{aligned}\delta\mathcal{F}_E &= \frac{AC}{2}\psi_{E_D^2} + \frac{BD}{2}\psi_E^2 + BC\psi_E\psi_{E_D}\\ &= \frac{1}{4}v^t\left[G^t\begin{pmatrix}0&1\\1&0\end{pmatrix}G - \begin{pmatrix}0&1\\1&0\end{pmatrix}\right]v\,,\end{aligned} \qquad (16)$$

where $\delta\mathcal{F}_E = \widetilde{\mathcal{F}}_E(\tilde{\psi}_E) - \mathcal{F}_E(\psi_E)$, $G = \begin{pmatrix}A&B\\C&D\end{pmatrix} \in SL(2,C)$, and $v = \begin{pmatrix}\psi_{E_D}\\ \psi_E\end{pmatrix}$.

Let us now consider the quantization of the expansions (14). First note that we have the consistency conditions

$$\psi_E = \sum_j \alpha_j^E \left(\sum_k \beta_k^E \psi_E^k \right)^j \Longleftrightarrow$$
$$x = \sum_k \beta_k^E \left(\sum_j \alpha_j^E x^j \right)^k, \tag{17}$$

which imply an infinite set of relations. Similar relations arise also for an arbitrary state described by a wave function ψ. In particular, expanding ψ in a given basis $\{\psi_j\}$, Eq. (14) generalizes to

$$\psi = \sum_j a_j \psi_j = \sum_j \alpha_j x^j \Longleftrightarrow$$
$$x = \sum_k b_k \psi_k = \sum_k \beta_k \psi^k, \tag{18}$$

implying an infinite set of relations which we denote by

$$D(\alpha, \beta) = 0. \tag{19}$$

Now observe that performing the second quantization

$$\psi \longrightarrow \hat{\psi} = \sum_j (\hat{a}_j \psi_j + \hat{a}_j^+ \overline{\psi}_j) \tag{20}$$

induces a quantization of the coefficients α_j's. Therefore, whereas the α_j's and β_k's enter in (19) as dual quantities, in the second quantization the α_j's only become operators. The important point is that Eq. (19), which is a manifestation of the x-ψ duality, suggests investigating whether there exists a quantization with the β_k's considered as operators. Therefore it is natural to consider

$$x \longrightarrow \hat{x} = \sum_k (\hat{\beta}_k \psi^k + \hat{\beta}_k^+ \overline{\psi}^k). \tag{21}$$

We now have two inequivalent dual pictures defined by (20) and (21), respectively. Whereas Eq. (20) corresponds to the second quantization of the wave function [associated to the Schrödinger equation (1)], Eq. (21) can be considered as the quantization of the coordinate [associated to Eq. (13), dual to Eq. (1)]. We note that as ψ takes complex values we can use the notation

$$\hat{x} = \sum_k (\hat{\beta}_k z^k + \hat{\beta}_k^+ \bar{z}^k). \tag{22}$$

This expression is conjectured in order to preserve the correspondence suggested by Eq. (18). The fact that this equation leads to the quantization of the coordinate should be further investigated. In particular, we remark that the structure of Eq. (22) resembles the expansion for the target coordinate in string theory. For the time being, we note that inverting $\psi = \psi(x)$ to $x = x(\psi)$ one obtains a description of geometrical quantities in terms of the wave function. Therefore we can think of the inversion method as a way to transfer quantum aspects directly to the coordinate, suggesting that (21) should play a role in quantizing geometry.

We have seen that both ψ_E and ψ_{E_D} enter on the same level, so that our formalism is manifestly modular invariant. An aspect which is related to this invariance is that there are quantum structures which may be described in the framework of monodromy transformations. For example, by (16) the $SL(2, \mathbf{Z})$ generators $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $|\psi_E^2|$ and ψ_E^2, respectively. These quantities correspond to the probability densities depending on if $\overline{\psi}_E \not\propto \psi_E$ or $\overline{\psi}_E \propto \psi_E$.

A feature of our approach is that it extends to higher dimensions. Furthermore, it may also be applied to the case of the Klein-Gordon equation (since the spinor components satisfy the Klein-Gordon equation, the construction applies to the fermionic case as well).

Let us first consider the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V(x) \right)\psi = E\psi, \tag{23}$$

where $\Delta = \sum_{k=1}^{D-1} \partial_{x_k}^2$. The way to find the generalization of Eq. (7) is to rewrite (23) in the form

$$\left(-\frac{\hbar^2}{2m}\partial_{x_k}^2 + V_k(x_k) \right)\psi = E\psi \tag{24}$$

for $k = 1, \dots, D-1$, where we have introduced the "effective potentials"

$$V_k(x_k) = \left[V(x) - \frac{\hbar^2}{2m\psi(x)} \sum_{j=1, j\neq k}^{D-1} \partial_{x_j}^2 \psi(x) \right]_{|x_{j\neq k} \text{ fixed}}. \tag{25}$$

Equation (24) is now seen as a second-order equation in the variable x_k, with $x_{j\neq k}$ considered as parameters for the effective potential V_k. Let $\psi_E^{(k)}$ and $\psi_{E_D}^{(k)}$ be linearly independent solutions of Eq. (24). Repeating the procedure considered in the one-dimensional case, where now for any k the integration is taken from x_{k0} to x_k keeping the other coordinate components fixed, we obtain

$$\frac{\sqrt{2m}}{\hbar} x_k(\psi_E^{(k)}) = \psi_E^{(k)^2} \frac{\partial \mathcal{F}_E^{(k)}}{\partial(\psi_E^{(k)^2})} - \mathcal{F}_E^{(k)} \tag{26}$$

for $k = 1, \dots, D-1$ and $(' \equiv \partial_{\psi_E^{(k)}})$

$$4\mathcal{F}_E^{(k)'''} + [V_k(x_k) - E]\left(\mathcal{F}_E^{(k)'} - \psi_E^{(k)}\mathcal{F}_E^{(k)''}\right)^3 = 0, \tag{27}$$

which is an ordinary differential equation for $\mathcal{F}_E^{(k)}(\psi_E^{(k)})$ once x_k in V_k is replaced with its functional dependence on $\psi_E^{(k)}$ given in (26). It is worth noticing that in the important case $V(x) = \sum_{j=1}^{D-1} f_j(x_j)$, the functional structure of $\mathcal{F}_E^{(k)}$ does not depend on the "parameters" $x_{j\neq k}$.

In the case of the Klein-Gordon equation, we rewrite $(\Box + m^2)\phi = 0$ in the form

$$[\partial^\mu \partial_\mu + V_\mu(x) + m^2]\phi = 0 \tag{28}$$

for $\mu = 0, ..., D - 1$, where we have introduced the effective potentials

$$V_\mu(x_\mu) = \left[\frac{1}{\phi(x)} \sum_{\nu=0,\nu\neq\mu}^{D-1} \partial^\nu \partial_\nu \phi(x) \right]_{|x_{\nu\neq\mu}\ \mathrm{fixed}} . \tag{29}$$

The important difference with respect to the case of the Schrödinger equation is that, as a consequence of its relativistic nature, the time derivative appears in the Klein-Gordon equation at the second order. This implies that the inversion formula also holds for the time component $x^0 = t$ and Eqs. (26) and (27) extend to the relativistic case with $k \in [1, D - 1]$ replaced by $\mu \in [0, D - 1]$.

Another manifestation of the statistical structure underlying the formalism is suggested by an analogy with $N = 2$ SYM. The point is that, in analogy with the role played by the scale Λ in Seiberg-Witten theory, we can interpret $\hbar$ as a parameter defining the scale of a statistical system. In particular, following the approach introduced in [3], we first note that for dimensional reasons

$$\frac{Kx}{\hbar} = \mathcal{G}(\tau), \tag{30}$$

where $K = \sqrt{2mE}$ and

$$\tau = \frac{\partial^2 \mathcal{F}_E}{\partial \psi_E^2} . \tag{31}$$

In this framework it makes sense to apply the operator

$$\hbar \partial_\hbar \tag{32}$$

to Eq. (30). We have

$$\beta \partial_\tau \mathcal{G}(\tau) = -\frac{Kx}{\hbar}, \tag{33}$$

where

$$\beta(\tau) = \hbar \partial_\hbar \tau . \tag{34}$$

Integrating (33) we obtain

$$x = \frac{\hbar}{\hbar_0} x_0 e^{-\int_{\tau_0}^{\tau} dy \beta^{-1}(y)}, \tag{35}$$

showing that the space coordinate has an anomalous dimension determined by the beta function (34). In this context we observe that the Heisenberg uncertainty principle depends on the scale

$$\Delta x \Delta p \geq \hbar = \hbar_0 + \text{corrections} . \tag{36}$$

We note that generalizations of the Heisenberg uncertainty principle have been discussed in the context of different approaches to quantum gravity in [5].

We observe that our approach sheds new light on the role of the dual wave function ψ_{D_E}. In this context $\mathcal{F}_E$ plays the crucial role as it can be seen as the analog of the Hamilton principal function. Actually, ψ_E and $\psi_{D_E} = \partial \mathcal{F}_E / \partial \psi_E$ play a similar role to x and p in Hamilton-Jacobi theory. The inversion formula Eq. (6) is the key starting point for this investigation [6].

Other aspects which merit further investigation concern the possible role of our construction in the framework of the stochastic approach to quantum mechanics [7], the many–particle systems, the case of coherent states, and geometric quantization. Here we limit ourself to observe that in the case of two-particle systems with central potential $V(r)$, one can find for r an expression similar to (6) with ψ_E and ψ_{E_D} replaced by χ_E and $\chi_{E_D} = \partial_{\chi_E} \mathcal{F}_E$, respectively [we are using the standard notation $\psi_E = Y_{lm}(\theta, \varphi)\chi_E(r)/r$]. This is a consequence of the fact that both χ_E and χ_{E_D} are in the kernel of the operator $\partial_r^2 + 2m(E - V)/\hbar^2 - l(l + 1)/r^2$, where $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

In conclusion, we observe that starting from the inversion formula we arrived at a statistical interpretation of the space coordinate opening the way for a possible understanding of the link between space-time structure and quantum theory. In particular, we stress that the inversion formula allows us to use

$$dx = \frac{\partial x}{\partial \psi} d\psi \tag{37}$$

for connecting geometrical and quantum concepts.

It is well known that after a Wick rotation of the time coordinate, the path-integral formulation of quantum mechanics resembles the partition function of a thermodynamical system. The appearance of the Legendre transform relating quantum and macroscopic quantities may clarify this relationship suggesting a possible thermodynamical interpretation of quantum mechanics whose implications may bring about a new deep understanding of the fundamental connection between geometry and quantum mechanics.

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