# Backreaction Problem for Cosmological Perturbations

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We derive the effective energy-momentum tensor for cosmological perturbations and prove its gauge invariance. The result is applied to study the influence of perturbations on the behavior of the Friedmann background in inflationary universe scenarios. We found that the back reaction of cosmological perturbations on the background can become important already at energies below the self-reproduction scale. [S0031-9007(97)02521-0]

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It is well known that gravitational metric perturbations treated as propagating on a curved "background space-time" have an effect on the evolution of this "background." This is due to the nonlinearity of the Einstein equations. A convenient way to describe the backreaction of fluctuations on the background is to consider the "effective" energy-momentum tensor (EMT) for these metric perturbations.

This problem has been studied by several authors in applications concerning gravity waves (see, e.g., [1–4], and references therein). One of the main puzzles needing to be solved is the problem of gauge invariance of the effective EMT. Namely, the effective EMT should be defined in a manner that the answer to the question "how important are perturbations for the evolution of a background?" does not depend on the choice of space-time coordinates (in other words, it should not depend on the gauge).

The issue of gauge invariance becomes critical when we attempt to analyze how gravitational waves and scalar metric perturbations produced in the early Universe influence the evolution of the background Friedmann-Robertson-Walker (FRW) universe. The procedure suggested by Isaacson [4] defines a gauge-invariant EMT for small-wavelength, high-frequency perturbations, and is not applicable in our case for the following reason. In order to get the invariant EMT following this prescription, one should average terms in the Einstein equations which are quadratic in the perturbations over time intervals bigger than the typical inverse frequency of perturbations. Obviously, it is assumed that the time scale characterizing the background is much bigger than the period of the perturbations. Since in the early Universe inhomogeneities with scales bigger than the horizon scale are frozen, it means that their typical period is much bigger than the cosmic time scale and the procedure cannot be used.

In this Letter we consider perturbations about a FRW manifold and show how to define a gauge-invariant EMT for metric perturbations which involves only spatial averaging on a hypersurface of constant time. This allows us to formulate the problem of backreaction of perturbations on the evolution of the background FRW universe in a coordinate-independent manner at every moment in time.

We apply our framework to a chaotic inflationary model. Given the spectrum of linear cosmological perturbations generated during inflation, we evaluate their effective EMT and find that backreaction becomes important already at energy scales lower than those at which the stochastic driving terms dominate. This may have important consequences for the dynamics of chaotic inflationary models.

There has been recent work on the backreaction of density inhomogeneities in cosmology. Futamase [5] considered the problem of backreaction in harmonic gauge. Seljak and Hui [6] reconsidered this issue using a different gauge but obtained differing results, thus highlighting the need for a gauge-independent analysis. A similar problem was also addressed by Buchert and Ehlers in the context of Newtonian cosmology [7].

This Letter is organized as follows: In Section 2 we formulate some useful properties of the diffeomorphism transformations. The backreaction problem is set up in Section 3, where we show how to formulate it in terms of gauge-invariant quantities only. Section 4 contains an application of our results to study the backreaction problem in the chaotic inflationary scenario.

*Diffeomorphism transformations.*—The gauge group of General Relativity is the group of diffeomorphisms. To define it we consider a smooth vector field $\xi^\alpha$ on the space-time manifold $\mathcal{M}$. The set of parametrized integral curves of $\xi^\alpha$ are given by solutions of the differential equations

$$\frac{d\chi^\alpha(\lambda)}{d\lambda} = \xi^\alpha[\chi^\beta(\lambda)], \tag{1}$$

($\lambda$ being an affine parameter) with initial conditions $\chi^\alpha(\lambda = 0) = x^\alpha$ for every $x^\alpha$. This induces a coordinate transformation on $\mathcal{M}$ (see also [8]):

$$x^\alpha \to \tilde{x}^\alpha = \chi^\alpha(\lambda = 1) = e^{\xi^\beta \frac{\partial}{\partial x^\beta}} x^\alpha$$
$$= x^\alpha + \xi^\alpha + \frac{1}{2}\xi^\alpha_{,\beta}\xi^\beta + \mathcal{O}(\xi^3), \tag{2}$$

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where $\xi$ should be considered small if we want to use a perturbative expansion in (2).

Now let us take two different points $P$ and $\tilde{P}$ of the manifold $\mathcal{M}$ having the same coordinate values $x_0^\alpha$ in the two distinct coordinate frames $x$ and $\tilde{x}$; that is $x_P^\alpha = x_0^\alpha$ and $\tilde{x}_{\tilde{P}}^\alpha = x_0^\alpha$. We want to express the value of an arbitrary tensor field $\tilde{\mathcal{Q}}_{\tilde{P}}$ at point $\tilde{P}$ in the coordinate system $\tilde{x}$ in terms of $\mathcal{Q}_P$ and its derivatives at point $P$ in the coordinate system $x$. The answer is well known and is given by the Lie derivative:

$$\begin{aligned}\tilde{\mathcal{Q}}(x_0) &= (e^{-\mathcal{L}_\xi}\mathcal{Q})(x_0)\\ &= \mathcal{Q}(x_0) - \mathcal{L}_\xi \mathcal{Q}(x_0) + \frac{1}{2}\mathcal{L}_\xi\mathcal{L}_\xi\mathcal{Q}(x_0)\\ &\quad + \mathcal{O}(\xi^3).\end{aligned} \tag{3}$$

This Lie operator obeys an important property, which we exemplify below in the case of the Einstein tensor $G$. We can express $G$ as a function of the metric and its derivatives:

$$G(x) \equiv G\left[\frac{\partial}{\partial x}, g(x)\right]. \tag{4}$$

Since the diffeomorphism transformation (3) does not effect the derivatives one can write

$$(e^{-\mathcal{L}_\xi}G)(x) = G\left[\frac{\partial}{\partial x}, (e^{-\mathcal{L}_\xi}g)(x)\right]. \tag{5}$$

Regarding $G(x)$ as a *functional* of the metric we can expand (5) in terms of functional derivatives and obtain, for example, the following property of the Lie derivative:

$$\mathcal{L}_\xi G(x) = \int d^4x' \frac{\delta G(x)}{\delta g(x')}\mathcal{L}_\xi g(x'), \tag{6}$$

where $\delta G(x)/\delta g(x')$ is the functional derivative of the Einstein tensor with respect to the metric. Formulas similar to (5) are true also for the EMT and, in fact, for arbitrary tensor fields which can be considered as local functionals of other tensor fields and their derivatives.

*Backreaction and gauge invariance.*—We consider a FRW universe with small perturbations. This means one can find a coordinate system $(t, x^i)$ in which the metric $(g_{\mu\nu})$ and matter fields $(\varphi)$, denoted for brevity by the collective variable $q^a \equiv (g_{\mu\nu}, \varphi)$, can be written as

$$q^a(t, x^i) = q_0^a(t) + \delta q^a(t, x^i), \tag{7}$$

where $q_0^a(t)$ depends only on the time variable and $|\delta q^a| \ll |q_0^a|$. It is also assumed that the spatial average of $\delta q^a$ over hypersurfaces $t = \text{const}$ with respect to the induced "homogeneous" part of the 3-metric vanishes.

The Einstein equations

$$G_{\mu\nu} - 8\pi T_{\mu\nu} := \Pi_{\mu\nu} = 0 \tag{8}$$

can be expanded in a functional power series in $\delta q^a$ about the background $q_0^a(t)$ if we treat $G_{\mu\nu}$ and $T_{\mu\nu}$ as functionals of $q^a$, namely,

$$\Pi(q_0^a) + \Pi_{,a}\delta q^a + \frac{1}{2}\Pi_{,ab}\delta q^a \delta q^b + \mathcal{O}(\delta q_0^3) = 0 \tag{9}$$

(omitting tensor indices). From now on we adopt DeWitt's condensed notation [9], i.e., assume that continuous variables $(t, x^i)$ are included with the field indices $a$, $b$ ..., so that, for instance, $q^{a'} \equiv q^a(t', x'^i)$ and $\Pi_{,a} \equiv \delta\Pi/\delta q^a|_{q_0}$ etc. In addition, the summation over repeated indices is understood to include integration over time and/or space.

To lowest order, the background $q_0^a(t)$ and the perturbations $\delta q^a$ satisfy, respectively, the equations

$$\Pi(q_0^a) = 0 \quad \text{and} \quad \Pi_{,a}\delta q^a = 0. \tag{10}$$

However, it is clear from (9) that to next order in $\delta q$ the perturbations also contribute to the evolution of the background homogeneous mode of the metric and matter fields $q_0^a$. To see this, we take the average of (9) over a $t = \text{const}$ hypersurface, and obtain the following "corrected" equations for the evolution of the background:

$$\Pi(q_0^a) = -\frac{1}{2}\langle\Pi_{,ab}\delta q^a \delta q^b\rangle, \tag{11}$$

where brackets $\langle\rangle$ denote averaging over constant time hypersurfaces. For instance, $\langle\Pi\rangle = \frac{\lim_{V\to\infty}\int_V \Pi d^3x}{\int_V d^3x}$. Since the integrand is quadratic in perturbation variables, the average is unchanged to second order under any gauge transformation. At first glance, it seems natural to identify the quantity on the right hand side of Eq. (11) with the effective EMT of perturbations which describes the backreaction of perturbations on the homogeneous background. However, this expression is not invariant with respect to diffeomorphism transformations and, for instance, does not vanish for "metric perturbations" induced in Minkowski space-time by a coordinate transformation.

Thus it is clear that if we want to clarify how important physical perturbations are for the background evolution we need a diffeomorphism independent (gauge-invariant) measure characterizing the strength of perturbations.

The coordinate transformations (2) induce diffeomorphism transformations (3) on $\delta q$ which, in linear order, take the form

$$\delta q^a \to \delta\tilde{q}^a = \delta q^a - \mathcal{L}_\xi q_0^a, \tag{12}$$

where $\langle\xi\rangle = 0$. To second order, the background variables $q_0^a$ are not gauge invariant either but change as

$$\begin{aligned}q_0^a \to \tilde{q}_0^a &= \langle e^{-\mathcal{L}_\xi}(q_0^a + \delta q^a)\rangle\\ &= q_0^a - \langle\mathcal{L}_\xi\delta q^a\rangle + \frac{1}{2}\langle\mathcal{L}_\xi^2 q_0^a\rangle.\end{aligned} \tag{13}$$

Let us write the metric for a perturbed flat FRW universe

$$\begin{aligned}ds^2 = {}& N^2(t)(1 + 2\phi)dt^2 - a^2(t)(B_{,i} - S_i)dx^i dt\\ & - a^2(t)[(1 - 2\psi)\delta_{ij} + 2E_{,ij} + F_{i,j}\\ & + F_{j,i} + h_{ij}]dx^i dx^j,\end{aligned} \tag{14}$$

where the 3-scalars $\phi$, $B$, $\psi$, $E$ characterize scalar perturbations, $S_i$ and $F_i$ are transverse 3-vectors, and $h_{ij}$ (gravity waves) is a traceless transverse 3-tensor [10].

Under a gauge transformation (12), the quantity $X^\mu \equiv [\frac{a^2(t)}{N^2(t)}(B - \dot{E}), -E_{,i} - F_i]$, with a "dot" denoting time derivative, changes as

$$X^\mu \to \tilde{X}^\mu = X^\mu + \xi^\mu. \tag{15}$$

This quantity will be treated formally as a 4-vector in Lie derivatives below. Using $X^\mu$ one can form gauge-invariant variables characterizing both background and linear perturbations: $Q = e^{\mathcal{L}_X} q$, that is,

$$\delta Q^a = \delta q^a + \mathcal{L}_X q_0^a \tag{16}$$

and

$$Q_0^a = q_0^a + \langle \mathcal{L}_X \delta q^a \rangle + \frac{1}{2} \langle \mathcal{L}_X^2 q_0^a \rangle. \tag{17}$$

It is easy to verify that the $\delta Q^a$ correspond to the set of Bardeen's gauge-invariant variables [10]. The $Q_0^a$ actually change under diffeomorphism transformations as

$$Q_0^a \to \tilde{Q}_0^a = Q_0^a + \frac{1}{2} \mathcal{L}_{\langle[\xi,X]\rangle} q_0^a, \tag{18}$$

where $[\xi, X]$ is the commutator of the vectors $\xi$ and $X$. For uncorrelated $\xi$ and $X$ we have $\langle[\xi, X]\rangle = 0$, and therefore the last term in (18) vanishes (see Ref. [11] for a detailed discussion of this term).

Our goal is to rewrite Eq. (11) in terms of quantities which are gauge invariant up to second order in perturbations. It is easy to see from identity (5) that if Einstein's equations are valid for the set of variables $q$, then

$$e^{\mathcal{L}_X} \Pi(q) = \Pi(e^{\mathcal{L}_X} q) = \Pi(Q) = 0. \tag{19}$$

Expanding (19) to second order in $\delta Q$ and taking the spatial average of the result yields

$$\Pi(Q_0) = -\frac{1}{2} \langle \Pi_{,ab} \delta Q^a \delta Q^b \rangle, \tag{20}$$

which is the desired gauge-invariant form of the backreaction equation. Note that in deriving (20) we made use of the equations of motion for $q$. Finally, Eq. (20) can be written as

$$G_{\mu\nu}(Q_0) = 8\pi [T_{\mu\nu}(Q_0) + \tau_{\mu\nu}(\delta Q)], \tag{21}$$

where

$$\tau_{\mu\nu}(\delta Q) \equiv -\frac{1}{16\pi} \langle \Pi_{,ab} \delta Q^a \delta Q^b \rangle \tag{22}$$

can be interpreted as the gauge-invariant effective EMT for perturbations. Therefore if we want to find out if the backreaction of perturbations is important we should compare $\tau_{\mu\nu}(\delta Q)$ with $T_{\mu\nu}(Q_0)$. Note that none of the terms in Eq. (21) depends on the specific coordinate system used to evaluate them.

To conclude this section, we will derive the effective EMT for scalar cosmological perturbations about a spatially flat FRW universe. Since the results do not depend on the gauge, we can calculate the EMT using a longitudinal gauge [10], in which

$$ds^2 = (1 + 2\phi) dt^2 - a^2(t)(1 - 2\psi) \delta_{ij} dx^i dx^j, \tag{23}$$

and the matter perturbation (taking matter to be a scalar field) is $\delta\varphi$. For many types of matter (scalar fields included) $T_{ij}$ is diagonal in linear order in $\delta q$, which implies that $\phi = \psi$ [10]. By evaluating the functional derivatives in (11) (see also [11]) one can derive the following expression for $\tau_{\mu\nu}$:

$$\begin{aligned}\tau_{00} = {} & \frac{1}{8\pi} [+12H \langle \phi \dot{\phi} \rangle - 3 \langle (\dot{\phi})^2 \rangle + 9a^{-2} \langle (\nabla\phi)^2 \rangle] \\ & + \frac{1}{2} \langle (\delta\dot{\varphi})^2 \rangle + \frac{1}{2} a^{-2} \langle (\nabla \delta\varphi)^2 \rangle \\ & + \frac{1}{2} V_{,\varphi\varphi}(\varphi_0) \langle \delta\varphi^2 \rangle + 2V_{,\varphi}(\varphi_0) \langle \phi \delta\varphi \rangle, \end{aligned} \tag{24}$$

$$\begin{aligned}\tau_{ij} = a^2 \delta_{ij} \Big\{ & \frac{1}{8\pi} [(24H^2 + 16\dot{H}) \langle \phi^2 \rangle + 24H \langle \dot{\phi} \phi \rangle + \langle (\dot{\phi})^2 \rangle + 4 \langle \phi \ddot{\phi} \rangle - \frac{4}{3} a^{-2} \langle (\nabla\phi)^2 \rangle] + 4\dot{\varphi}_0^2 \langle \phi^2 \rangle + \frac{1}{2} \langle (\delta\dot{\varphi})^2 \rangle \\ & - \frac{1}{6} a^{-2} \langle (\nabla\delta\varphi)^2 \rangle - 4\dot{\varphi}_0 \langle \dot{\delta\varphi} \phi \rangle - \frac{1}{2} V_{,\varphi\varphi}(\varphi_0) \langle \delta\varphi^2 \rangle + 2V_{,\varphi}(\varphi_0) \langle \phi \delta\varphi \rangle \Big\}, \end{aligned} \tag{25}$$

where $H = \dot{a}/a$ is the Hubble parameter and $\tau_{0i} = \tau_{ij} = 0$ $(i \neq j)$.

*Backreaction in stochastic inflation.*—As an application of the formalism developed in the previous sections, we will evaluate the order of magnitude of backreaction effects in the chaotic inflationary scenario [12,13], for simplicity taking a massive scalar field as the inflaton. In this model, quantum fluctuations of the scalar field $\varphi$ certainly dominate the dynamics of the background when the field is above the self-reproduction scale $\varphi_{sf} \sim m^{-1/2}$ (in Planck units), and space on scales of the particle horizon is completely inhomogeneous, consisting of many bubble universes. It is usually supposed that in spatial regions where the scalar field at some point drops below $\varphi_{sf}$, the evolution proceeds classically and the metric fluctuations generated are not very important for the evolution of the homogeneous background. We will show below that this is not really the case.

In a chaotic inflationary universe scenario, linear perturbations on a fixed comoving scale $k$ are completely specified by the function $\phi_k$ (for a review, see [10]). This is due to the fact that $\psi = \phi$ and that the metric and matter perturbation variables $\phi$ and $\delta\varphi$ are anticorrelated for $ka \ll H$, i.e., $\delta\varphi_k \simeq -\varphi_0\phi_k$. Hence, all terms in the effective energy-momentum tensor $\tau_{\mu\nu}$ can be expressed through the various correlators of $\phi_k$. The amplitudes of $\phi_k$ are known from the theory of linear cosmological perturbations. Using the results for $\phi_k$ valid during inflation [10] we obtain, for instance, the regularized correlator

$$\begin{aligned}\langle\phi^2\rangle &= \int_{k_i}^{k_t} \frac{dk}{k}|\delta_k^{\phi}|^2 \\ &= \frac{m^2}{32\pi^4\varphi_0^4(t)} \int_{k_i}^{k_t} \frac{dk}{k}\left[\ln\frac{H(t)a(t)}{k}\right]^{-2} \\ &\sim m^2\frac{\varphi_0^6(t_i)}{\varphi_0^4(t)}\,,\end{aligned} \tag{26}$$

where $t$ denotes physical time, $t_i$ is the time when inflation started, and the inflaton potential is $V = 1/2m^2\varphi^2$. The IR and UV physical cutoffs $k_i$ and $k_t$ are given, respectively, by the scale of the largest wavelength perturbation (created when inflation started at time $t_i$), i.e., $k_i = H(t_i)a(t_i)$, and by the scale $k_t = H(t)a(t)$ of the shortest classical perturbation, which is just the scale of the Hubble distance.

It can be checked that the main contribution to the EMT of cosmological perturbations $\tau_{\mu\nu}$ comes from terms proportional to the above correlator. Therefore one finds that at the end of inflation (when $\varphi_0 \sim 1$) the energy density of perturbations is about

$$|\tau_{00}| \sim m^4[\varphi_0(t_i)]^6. \tag{27}$$

Comparing the above result (27) with the background energy density at the same moment of time, we conclude that if at the beginning of inflation

$$\varphi_0(t_i) > \varphi_{br} \sim m^{-1/3}, \tag{28}$$

then backreaction becomes important before the end of inflation ($\varphi_o \sim 1$).

It is important to note that $\varphi_{br}$ is smaller than the value $\varphi_{sf} \sim m^{-1/2}$ when stochastic source terms from quantum fluctuations start to dominate. A more detailed discussion of backreaction will be the subject of a forthcoming publication [11].

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