Fluctuation-Induced Transport in a Periodic Potential: Noise versus Chaos

In a recent Letter in this journal Hondou and Sawada describe symmetry breaking in the dynamics of an overdamped particle subject to a force which is chaotic [1]. They analyze a map

$$x_{n+1} = x_n - V'(x_n) - \eta_n, \quad (n = 0, 1, 2, \ldots), \qquad (1)$$

where $V(x + L) = V(x)$ is periodic potential, and η_n is noise given either by the tent map $\eta_{n+1} = -2|\eta_n| + 1/2$, or by the Bernoulli shift $\eta_{n+1} = 2\eta_n - 1/2\,\mathrm{sgn}(\eta_n)$. Both maps are chaotic and have uniform invariant measures $\rho(x) = 1$ on the interval $x \in [-1/2, 1/2]$. Continuous analogs of (1) $(x_{n+1} - x_n \to dx/dt)$ have been extensively studied recently for an asymmetric potential $V(x)$ in the context of fluctuation-induced transport [2]. The occurrence of a net drift of the particle J was observed in the case where the driving noise has a finite correlation time or the system is driven by an additional sinusoidal force. In [1] it was found that even in the case of a symmetric potential $V(x) = V(-x)$, the drift $J \equiv \lim_{n\to\infty} x_n/n \neq 0$ if η in (1) is generated by the tent map, although $J = 0$ for the noise generated by the Bernoulli shift. In [1] it was stated that this effect is "a clear-cut result of the hidden order of the chaotic system." It was also stated that no "ordinary statistical quantities, such as correlation function of chaotic noise (can) explain the difference of the overall dynamics in multistable systems."

We contend that the onset of current *can* be understood in terms of statistical quantities, and that this phenomenon arises if a particle is driven by any noise (including that due to fluctuations in a system with an infinitely large number of degrees of freedom) as long as the noise is *dynamically asymmetric* [3]. For such noise, realizations $\{\eta_n, \eta_{n+1}, \ldots\}$ and $\{-\eta_n, -\eta_{n+1}, \ldots\}$ which differ by the signs of η have different probabilities. Even if the probability density is symmetrical $\rho(\eta_i) = \rho(-\eta_i)$, the joint probability densities are not, $\rho(\eta_1, \ldots, \eta_m) \neq \rho(-\eta_1, \ldots, -\eta_m)$, and even though $\langle \eta_n^{2k+1} \rangle = 0$ for all $k = 0, 1, 2, \ldots$, some of its odd *dynamical* correlators $\langle \eta_{n_1} \ldots \eta_{n_{2k+1}} \rangle$ are not equal to zero [4(a)]. The tent map investigated in [1] does not have inversion symmetry $\eta_n \to -\eta_n$, and for the corresponding noise certain odd dynamical correlators are finite, such as $\langle \eta_n^2 \eta_{n+1} \rangle / \langle \eta_n^2 \rangle = -1/4$. Therefore there is a net drift. In contrast, for noise generated by the Bernoulli shift all odd-order correlators are equal to zero, and therefore there is no net drift.

We expect a net drift to arise generically in the case of a noise-driven particle in a symmetric periodic potential $V(x)$ when the dynamical response of the particle has a memory effect and the noise is dynamically asymmetric. For a retarded response, the average force exerted by the noise on the particle depends on the *evolution* of the noise, not only on the instantaneous values of η_n [4], and this dependence is nonlinear for a nonlinear $V'(x)$ [which is always true in the case of a periodic $V'(x)$]. Since opposite-sign sequences of η_n have different probabilities, the overall average force is not equal to zero, and therefore there arises a net drift. If there were no memory effects in the particle dynamics, the average force would be equal to zero for an even distribution $\rho(\eta_n)$, as well as for a dynamically symmetric noise.

A mathematical insight can be gained from the analysis of a case where, on the average, the noise in (1) is stronger than the potential $V(x)$. In this case, to the lowest approximation $x_n^{(0)} = x_0 - \sum_{i=0}^{n-1} \eta_i$, and the drift is

$$J \equiv -\langle V'(x_n) \rangle \approx \left\langle V''(x_n^{(0)}) \sum_{m=0}^{n-1} V'(x_m^{(0)}) \right\rangle, \qquad (2)$$

where the right-hand side (rhs) of (2) is independent of n and x_0 for large n. Since $V''(-x) = -V'(x)$, and $V''(-x) = V''(x)$, the expression in the rhs of (2) is equal to zero for a symmetric noise where the sequences $\{\eta_1, \eta_2, \ldots\}$ and $\{-\eta_1, -\eta_2, \ldots\}$ are equally probable, but is finite for an asymmetric noise, as we have verified numerically.

The existence of nonvanishing odd-order correlations is a ubiquitous feature of various types of noise, the simplest example being noise produced by a noncentrosymmetric oscillator coupled to a thermal bath. Therefore occurrence of drift in a symmetric periodic potential would be expected to be a common feature of noise-driven systems, but by no means just "retrieves the deterministic nature of chaos," as stated in [1].

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