

Infinite-Cluster Geometry in Central-Force Networks

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We show that the infinite percolating cluster (with density P_∞) of central-force networks is composed of a fractal stress-bearing backbone (P_B) and rigid but unstressed “isostatic ends” which occupy a finite volume fraction of the lattice (P_I). Near the rigidity threshold p_* there is then a *first-order transition* in $P_\infty = P_I + P_B$, while P_B is *second order* with exponent β' . A new mean-field theory suggests $\beta'_{\text{mf}} = 1/2$, while simulations of triangular lattices give $\beta'_t = 0.25 \pm 0.03$. [S0031-9007(97)02383-1]

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The connectivity-percolation geometry has become a paradigm in the study of disordered systems [1], especially close to the percolation critical point. However, in many problems involving mechanical properties, such as the mechanical properties of granular media [2], glasses [3], and gels [4], the connectivity-percolation geometry does not apply. In systems such as these, in which *central forces* are of primary importance, the infinite cluster must be *multiply connected* in order to transmit stress and hence to support any mechanical property. Here we present a comprehensive analysis of the geometry of the infinite cluster in this “rigidity-percolation” problem.

Early work on the stress-bearing paths in central-force systems relied on a direct solution to the force equations for model systems [5–8], such as lattices of Hookian springs. More recently, a more accurate method which finds the backbone [9] and infinite-cluster [10] geometry directly has been developed. There has also been progress in the development of mean-field theories, using both continuum [11] and Cayley-tree models, and we report on the latter here. Using a combination of these new techniques a novel picture of rigidity percolation has emerged. In particular, we show that the infinite cluster is composed of a stress-carrying backbone and rigid (but unstressed) “isostatic ends” [see Fig. 1(a)]. The backbone is *fractal*, and this is the reason that the *elastic constants undergo a second-order transition* on approach to the rigidity transition [5–9]. However, the isostatic ends occupy a finite volume fraction of the lattice, and for this reason there is a *first-order transition* in P_∞ at p_* ; because of the fractal backbone, there is a length which diverges with a nontrivial exponent, ν , at the rigidity threshold. This exponent is different than the connectivity-percolation correlation-length exponent. The number of “red” or cutting bonds scales with exponent $1/\nu$ at the rigidity threshold.

Consider a triangular lattice composed of nodes connected by Hookian springs. Consider attaching rigid “bus bars” on two sides of the lattice, and then removing $1 - p$ of the sites (or bonds) from the lattice. Upon application of an external stress (e.g., a tensile stress) to the lattice, we find results such as that presented in Fig. 1(a). In this figure, the *black* bonds carry stress, while the *green* bonds are rigid but do not carry stress. The *red* bonds carry stress and are critical in the sense that if any one of them is removed, the lattice can no longer support the applied stress (they are “cutting” bonds) [12]. When one of the “red” bonds is removed, the infinite rigid cluster collapses like a “house of cards” into a large number of smaller rigid clusters. This is the basic reason why the rigidity transition is first order. Figure 1(a) was found using a new “graph-theory” method [13] which enables us to find the rigidity-percolation geometry exactly and which has been described elsewhere [9,10]. We plot the volume fraction of backbone (red plus black), isostatic ends (green), and infinite-cluster bonds (red plus black plus green) as a function of sample size in Fig. 1(b) for both site- and bond-diluted networks. It is seen in this figure that, although the volume fraction of “isostatic” bonds is rather small (roughly 0.1), it is constant, indicating that the infinite-cluster probability undergoes a first-order jump (or has an exceedingly small exponent) at the rigidity threshold. The stressed backbone, however, shows a nontrivial scaling, so that $n_B \sim L^{D_B}$, with $D_B = 1.78 \pm 0.02$ as found previously [9,10]. Note that a fit to the infinite-cluster data of Fig. 1(b) suggests that the infinite cluster is fractal, as claimed by Jacobs and Thorpe [10]. However, by separating the isostatic ends from the backbone bonds as shown in Fig. 1(b), the weakly first-order character of P_∞ is evident. Thus in the thermodynamic limit, we predict

$$P_\infty = a + b(p - p_*)^{\beta'}, \quad (1)$$

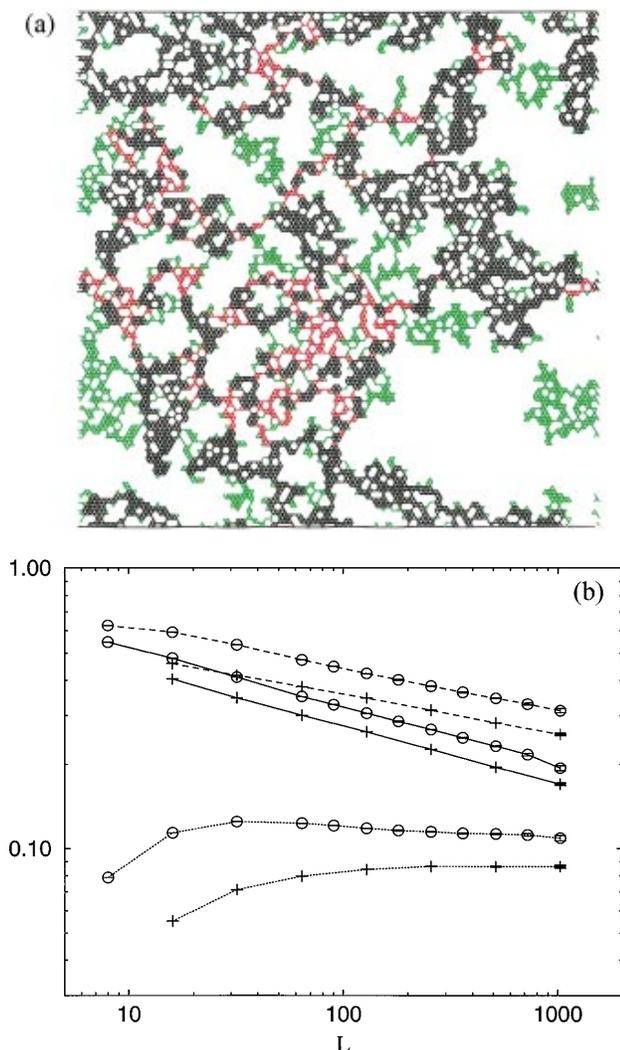


FIG. 1. The geometry of the infinite rigid cluster (a) (color) An infinite rigid cluster at the rigidity threshold of a site-diluted triangular lattice of size $L = 128$. Backbone (stressed) bonds are black, isostatic ends (rigid but unstressed bonds) are green, and cutting bonds (if one of them is removed the backbone collapses) are red. (b) P_∞ (long dashes); P_B (solid lines), and P_I (short dashes) for site- (plusses) and bond- (circles) diluted triangular lattices. The calculations were done at the percolation point for each lattice size and are averaged over roughly $10^8/L^2$ configurations.

where for the triangular lattice, the backbone exponent $\beta'_i = 0.25 \pm 0.03$. Here we have used the result, $2 - \beta'_i/\nu = D_B$, with $\nu = 1.16 \pm 0.03$ from our previous calculations [9,10]. For the site-diluted triangular lattice the first-order jump in P_∞ , $a_{sd}^i = 0.086 \pm 0.005$, while for the case of bond dilution $a_{bd}^i = 0.11 \pm 0.02$ [see Fig. 1(b)]. Our mean-field theory uses exact “constraint counting” (see below) on trees. We assign each node of a lattice g degrees of freedom and coordination z . On the triangular lattice treated in the previous paragraph, each node has two degrees of freedom and $z = 6$ for the pure lattice. If the nodes of a triangular lattice were *extended*

objects or *bodies* instead of being pointlike “joints,” then each node would have an additional rotational degree of freedom. Cases of physical interest are then

$$g = 1 \quad \text{for connectivity percolation,} \quad (2)$$

$$g = d \quad \text{for a joint,} \quad (3)$$

$$g = d + \frac{d(d-1)}{2} \quad \text{for a body,} \quad (4)$$

where d is the dimensionality of the embedding space. Knowledge of g and z is enough to write down the simplest constraint-counting mean-field theory [3]. Consider bond dilution with probability p that a bond is present. Present bonds restrict the motion of the nodes, and hence they impose “constraints” on the degrees of freedom of the lattice. We assume that all of the constraints are “independent” (a “dependent” bond can be removed without causing any reduction in the size of the rigid clusters), then the number of degrees of freedom that remain unconstrained (the “floppy” modes per site f) is approximated by

$$f = g - pz/2 \quad \text{for } p < p_* = 2g/z. \quad (5)$$

For $p > p_*$, $f = 0$ in this approximation. This mean-field theory has been useful in the study of glasses [3], but has yielded little information about P_∞ , P_B , and P_I . More recently, a continuum mean-field theory has been developed [11], which focuses on P_∞ . That theory indicates that the infinite-cluster probability undergoes a first-order transition at the rigidity threshold, but the connection with the key parameters g and z is unclear. In addition, in that work a pathological lattice model (a square lattice with random diagonals) was suggested as a paradigm for the rigidity transition. We show later why that model is anomalous.

We now present our Cayley-tree model for rigidity percolation, which provides a complete mean-field model for arbitrary g and z . The trees have coordination number z but, as usual in tree models, the key results come from consideration of one branch of a tree [see Fig. 2(a)]. Consider site-diluted trees which are grown from a rigid boundary at infinity. Building inward from this boundary, we keep track of the number of degrees of freedom of a node with respect to the boundary. Rigidity can only be transmitted to higher levels of the tree if there are enough rigid bonds present to offset the g degrees of freedom of a newly added node. For connectivity percolation only one bond is needed. If a node is added to a $g = 2$ tree, two bonds are needed to offset the two degrees of freedom of the added node. In general, if the nodes of the tree have g degrees of freedom, rigidity is transmitted to the next level of the tree provided the node is occupied, and provided at

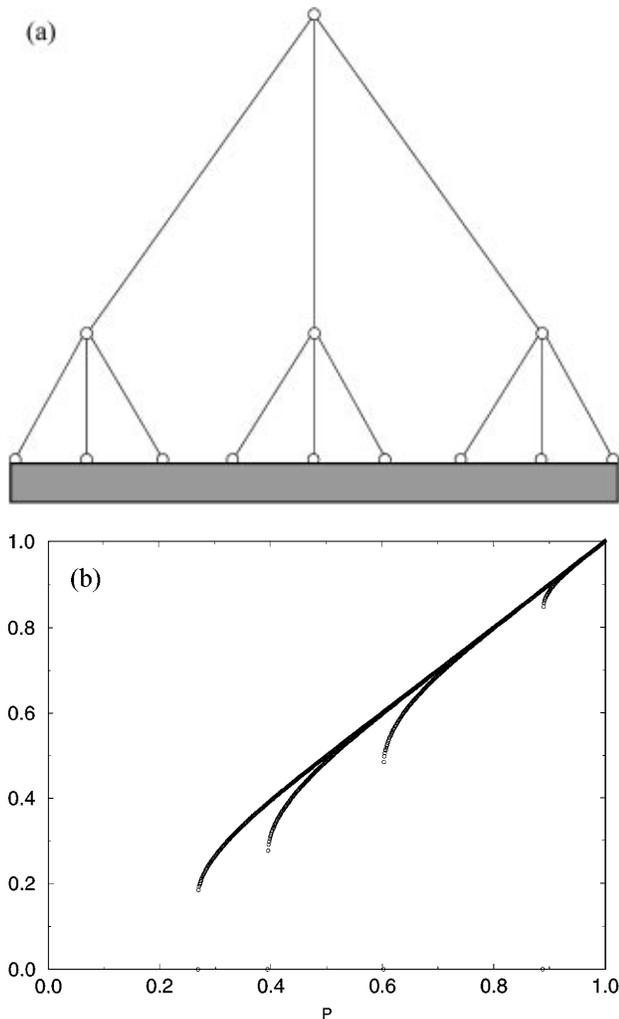


FIG. 2. (a) One branch of a $z = 4$ Cayley tree. The lowest-level branches of the tree are attached to a rigid border. (b) P_∞ for site-diluted trees with $g = 2$ and, starting with the rightmost curve, $z = 4, 6, 9, 13$.

least g of the lower-level nodes are rigid. This gives the recurrence relation

$$T_\infty^{l+1} = p \sum_{k=g}^{z-1} \binom{z-1}{k} (T_\infty^l)^k (1 - T_\infty^l)^{z-1-k}, \quad (6)$$

where T_∞^l is the probability that a site which is l levels from the rigid border (which is level 0) is rigid with respect to the border. For $g = 1$ this reproduces the familiar model for connectivity percolation [14], while for $g > 1$ it is mathematically equivalent to so-called “bootstrap” percolation [15] (note that for lattices with internal loops this equivalence does not usually apply). If we take the thermodynamic limit (very large l), Eq. (6) iterates to a steady-state solution, which we call T_∞ . Finally we find P_∞ from

$$P_\infty = p \sum_{k=g}^z \binom{z}{k} (T_\infty)^k (1 - T_\infty)^{z-k}. \quad (7)$$

The results for T_∞ for several values of g and z are presented in Fig. 2(b). It is seen from this figure that

for $g = 2$, the rigidity transition is first order; we show elsewhere that this conclusion holds for all $g > 1$ [16]. An exact solution is straightforward for the case $z = 4$, $g = 2$, where [from Eq. (6)]

$$T_\infty = p[T_\infty^3 + 3T_\infty^2(1 - T_\infty)], \quad (8)$$

which has the trivial solution $T_\infty = 0$ and the nontrivial solutions

$$T_\infty = \frac{3 \pm \sqrt{9 - 8/p}}{4}. \quad (9)$$

In order to ensure that $T_\infty = 1$, when $p = 1$, take the positive solution in Eq. (9). The square root in Eq. (9) becomes imaginary at $p_* = 8/9$. As $p \rightarrow p_*$, Eq. (9) yields

$$T_\infty = \frac{3}{4} + c(p - p_*)^{1/2}. \quad (10)$$

The key feature of this equation is that T_∞ is first order, and there is a *square-root singularity* superimposed upon the first-order transition. We find a similar behavior for all g and z , and so we write in general

$$P_\infty = a + b(p - p_*)^{1/2}, \quad (11)$$

where a , b , and p_* are dependent on g and z . Notice that this is of the same form as Eq. (1) for the triangular lattice. We thus argue that in general the form Eq. (1) is correct, with the mean-field backbone-exponent $\beta'_{mf} = \frac{1}{2}$. Full details of the Cayley-tree result will be published elsewhere [16].

Because of the fractal backbone, there is a diverging length, and the scaling properties are controlled by this diverging length. As shown in previous calculations this “correlation” length diverges with exponent $\nu = 1.16 \pm$

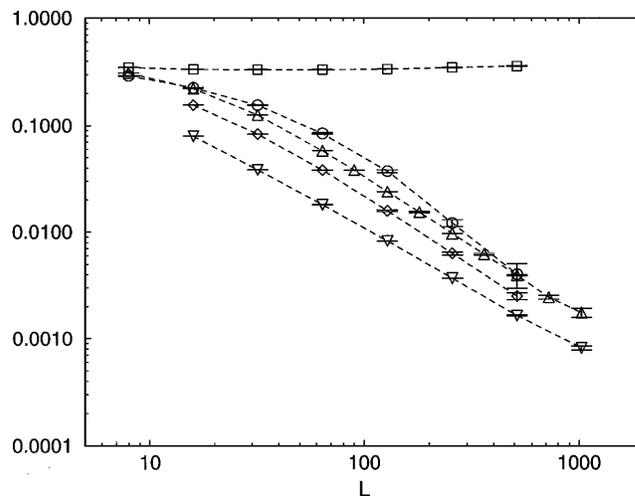


FIG. 3. The volume fraction of red bonds P_r at the rigidity threshold for the triangular lattice with site dilution (inverted triangles) and bond dilution (triangles); the “random diagonal” model (see text) with $q = 0$ (squares—this is the anomalous model of Fig. 4), $q = 0.1$ (circles) and $q = 0.4$ (diamonds).

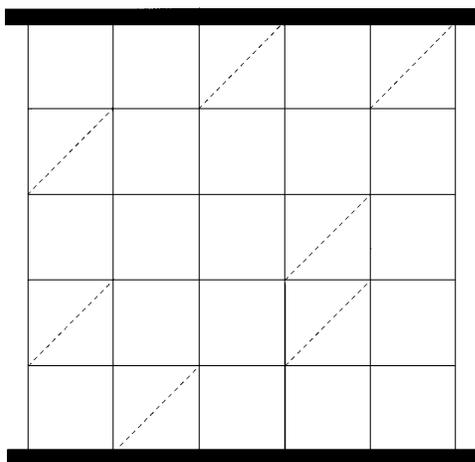


FIG. 4. The square lattice with random diagonals and no removed horizontal or vertical bonds, so $q = 0$ (see text).

0.03 [9,10] at the rigidity threshold in two dimensions. Coniglio's relation then applies to this problem [12], so that we expect n_r , the number of red or "cutting" bonds, to diverge as $L^{1/\nu}$ at the rigidity threshold. We find that to be the case for both the site- and bond- diluted triangular lattices (see Fig. 3), from which we find $P_r \sim n_r/L^2 \sim L^{x-2}$, with $x = 0.86 \pm 0.02 = 1/\nu$ confirming our previous result [9].

Finally, we analyze and extend a simple model for gels, which has been developed by Obukhov [11] to provide an illustration of a first-order rigidity transition (see Fig. 4—the boundary conditions do not change the conclusions). The random diagonals are present with probability p_d , and all that is required to make the lattice rigid is one diagonal present in every row of the square lattice. The probability that an infinite-cluster exists is then $P_+ = [1 - (1 - p_d)^L]^L$, from which we find $p_{d*} \sim \ln L/L$. However, the resulting infinite cluster contains the whole lattice, the rigid backbone is extensive, and so is the number of cutting bonds (see Fig. 3 for a calculation of the cutting bonds). This is inconsistent with the result Eq. (1) and also with the presence of a diverging length at the rigidity transition. However, the model of Fig. 4 is atypical, as can be seen by considering a generalized model in which we randomly add the diagonals (with probability p_d) to a square lattice whose bonds have been diluted with probability q (Obukov's model is $q = 0$), while if $q = 1 - p_d$ this model is equivalent to the bond-diluted

triangular lattice. We find that even for a small amount of dilution, e.g., $q = 0.10$, the rigidity transition returns to the behavior characteristic of the isotropic triangular case (see Fig. 3). We find that for sufficiently large lattice sizes, the universal behavior Eq. (1) holds for any finite $q < 0.5$, and that the "fully first-order" transition (i.e., a first-order backbone) only occurs in the special case of a perfect square lattice (or other "marginal" regular lattices) with randomly added diagonals. We have found similar pathological cases on Cayley trees.

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