Modular Invariance and Structure of the Exact Wilsonian Action of $N = 2$ Supersymmetric Yang-Mills Theory

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We construct modular invariants on $\mathcal{M}_{\mathrm{SU(2)}}$, the moduli space of quantum vacua of $N = 2$ supersymmetric Yang-mills theory with gauge group SU(2). We also introduce the nonchiral function $\mathcal{K}(A, \overline{A}) = 2\alpha\pi e^{\varphi_{\mathrm{SW}} - \varphi/2}$, where $e^{\varphi_{\mathrm{SW}}}$ is the Seiberg-Witten metric, e^{φ} is the Poincaré metric on $\mathcal{M}_{\mathrm{SU(2)}}$ and α is a regularization scheme-dependent constant. It turns out that $\mathcal{K}(A, \overline{A})$ has all the expected properties of the next to leading term in the Wilsonian effective action $S[A, \overline{A}]$ whose modular properties are considered in the framework of the dimensional regularization. [S0031-9007(97)02373-9]

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The exact results about $N = 2$ supersymmetric Yang Mills theory (SYM) obtained by Seiberg and Witten [1] concern the low-energy Wilsonian effective action with at most two derivatives and four fermions. In the SU(2) case, the u-moduli space of quantum vacua is $\mathcal{M}_{\mathrm{SU(2)}}$, the Riemann sphere with punctures at $u = \infty$, $u = \pm\Lambda^2$. In [2] results in [1] have been derived from first principles [3,4]. In particular, the T^2 symmetry $u(\tau + 2) = u(\tau)$, which rigorously follows from the asymptotic analysis together with the relation [3]

$$u = \pi i (\mathcal{F} - a\partial_a \mathcal{F}/2),\tag{1}$$

and the fact that

$$\overline{u(\tau)} = u(-\overline{\tau}), \quad u(\tau + 1) = -u(\tau),\tag{2}$$

uniquely fix the monodromy group Γ to be $\Gamma(2)$. The basic observation is that, for real values of u, we have the symmetry $u(\tau) = u(-\overline{\tau})$ which essentially fixes Γ. The reason is that by (1) and Im $\tau > 0$ (except for the singularities where Im $\tau = 0$), u can be seen as uniformizing coordinate. Therefore, $\mathcal{M}_{\mathrm{SU(2)}} \cong H/\Gamma$ where H is the upper half plane (the τ-moduli space; see [5]). This is equivalent to $u(\gamma \cdot \tau) = u(\tau)$ with $\gamma \in \Gamma$. It follows that there are curves C in the fundamental domains in H such that for $\tau \in C$ one has $\gamma \cdot \tau = -\overline{\tau}$. This reasoning together with a proper use of $u(\tau + 1) = -u(\tau)$ essentially implies the results in [2].

In [1] it has been emphasized that the metric,

$$ds^2 = \mathrm{Im}(\partial_a^2 \mathcal{F})|da|^2 = |\partial_u a|^2 \mathrm{Im}(\partial_a^2 \mathcal{F})|du|^2,\tag{3}$$

is at the heart of the physics. The natural framework to investigate its properties is uniformization theory [2,3,5].

In this Letter we use basic geometric structures of $\mathcal{M}_{\mathrm{SU(2)}}$ to derive a modular invariant quantity which fulfills all the expected properties of the next to leading term in the Abelian Wilsonian effective action [6–9].

Let us now recall the metric introduced in [5]. Let $H = \{w | \mathrm{Im}\, w > 0\}$ be the upper half plane endowed with the Poincaré metric $ds_P^2 = (\mathrm{Im}\, w)^{-2}|dw|^2$. Since

$\tau = \partial_a^2 \mathcal{F}$ is the inverse of the map uniformizing $\mathcal{M}_{\mathrm{SU(2)}}$, it follows that the positive definite metric,

$$ds_P^2 = \frac{|\partial_a^3 \mathcal{F}|^2}{(\mathrm{Im}\,\tau)^2}|da|^2 = \frac{|\partial_u \tau|^2}{(\mathrm{Im}\,\tau)^2}|du|^2 = e^{\varphi}|du|^2,\tag{4}$$

is the Poincaré metric on $\mathcal{M}_{\mathrm{SU(2)}}$. This implies that φ satisfies the Liouville equation $\varphi_{u\overline{u}} = e^{\varphi}/2$.

We now show that the Seiberg-Witten metric

$$e^{\varphi_{\mathrm{SW}}} = |a'|^2 \mathrm{Im}\,\tau,\tag{5}$$

where $' \equiv \partial_u$, can be written in terms of the Poincaré metric e^{φ}. To this end we first summarize a few facts. A crucial role in the theory is played by the Picard-Fuchs equations [10,11]; in particular, in the SU(2) case we have the (reduced) uniformizing equation [3,10]

$$[4(u^2 - \Lambda^4)\partial_u^2 + 1]\psi = 0,\tag{6}$$

satisfied by a_D and a, implying that

$$(\Lambda^4 - \mathcal{G}^2)\partial_a^2 \mathcal{G} + \frac{a}{4}(\partial_a \mathcal{G})^3 = 0,\tag{7}$$

where $u = \mathcal{G}(a)$. By (1) and (7) we have [5]

$$\partial_a^3 \mathcal{F} = \frac{\pi^2 (a\partial_a^2 \mathcal{F} - \partial_a \mathcal{F})^3}{16[\Lambda^4 + \pi^2 (\mathcal{F} - a\partial_a \mathcal{F}/2)^2]}.\tag{8}$$

Furthermore, by [1]

$$a_D = \partial_a \mathcal{F} = \frac{\sqrt{2}}{\pi}\int_{\Lambda^2}^u \frac{dx\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}},$$
$$a = \frac{\sqrt{2}}{\pi}\int_{-\Lambda^2}^{\Lambda^2}\frac{dx\sqrt{x - u}}{\sqrt{x^2 - \Lambda^4}},\tag{9}$$

we have $a(-\Lambda^2) = -i4\Lambda/\pi$ and $a(\Lambda^2) = 4\Lambda/\pi$. It follows that the initial conditions for the second-order differential equation (7) are $\mathcal{G}(-i4\Lambda/\pi) = -\Lambda^2$ and $\mathcal{G}(4\Lambda/\pi) = \Lambda^2$ and by (1)

$$\mathcal{F}(a) = \frac{2i}{\pi}a^2 \int_{4\Lambda/\pi}^a db\,\mathcal{G}(b)b^{-3} - \frac{\pi i}{16}a^2.\tag{10}$$

Now observe that by (8)

$$\partial_u \tau = \frac{1}{2\pi i a'^2(\Lambda^4 - u^2)}. \tag{11}$$

Since $e^\varphi = |\partial_u \tau|^2/(\mathrm{Im}\,\tau)^2$, we can rewrite Eq. (11) as

$$e^{-\varphi/2} = 2\pi |a'|^2 \, |u^2 - \Lambda^4| \mathrm{Im}\,\tau. \tag{12}$$

We stress that, as observed in [3] for the term $|u^2 - \Lambda^4|$ in the uniformizing equation, also in (12) it should be considered as a $(-3/2, -3/2)$ differential on $\mathcal{M}_{\mathrm{SU(2)}}$. This ensures the covariance of Eq. (12). By (5) we have

$$e^{\varphi_{\mathrm{sw}}} = \frac{e^{-\varphi/2}}{2\pi |u^2 - \Lambda^4|}. \tag{13}$$

Now observe that by (1) we have

$$a' = \frac{2}{\pi i a(\hat\tau - \tau)}, \tag{14}$$

where $\hat\tau = a_D/a$. Therefore (12) is equivalent to

$$e^{-\varphi/2} = \frac{8}{\pi} \frac{|u^2 - \Lambda^4|}{|a|^2 \, |\hat\tau - \tau|^2} \mathrm{Im}\,\tau. \tag{15}$$

In [6] it has been shown that the next to leading term which contributes to the Abelian Wilsonian effective action $S[A,\overline{A}]$ is $\int d^4x \, d^4\theta \, d^4\overline\theta \, \mathcal{K}(A,\overline{A})$ where $\mathcal{K}(A,\overline{A})$ is a modular invariant real analytic function of the $N = 2$ U(1) vector multiplet A. Therefore, up to this order $S[A,\overline{A}]$ has the form $\int d^4x \, d^4\theta \, d^4\overline\theta \, \mathcal{K}(A,\overline{A}) + \frac{1}{4\pi} \mathrm{Im}[\int d^4x \, d^4\theta \, \mathcal{F}(A)]$. In Ref. [7] de Wit, Grisaru, and Roček were able to prove that asymptotically

$$\mathcal{K}(A,\overline{A}) \sim c \ln \frac{A}{\Lambda} \ln \frac{\overline{A}}{\Lambda}, \tag{16}$$

where c is a constant. Furthermore, the one-instanton contribution to $\mathcal{K}$ has been obtained by Yung [8]. We observe that one should expect that the logarithmic singularities at $u \to \infty$, such as the asymptotic behavior in (16) be the unique singularities in $\mathcal{K}$ as possible singularities at $u \neq \infty$ will spoil the physical meaning of the quantum moduli space. In particular, $\mathcal{K}$ should be regular where monopoles or dyons become massless (see also [8]). Actually, it seems that the only possible way in order to not change the physical picture in [1] is that $\mathcal{K}$ be vanishing at these points. Let us illustrate this aspect by recalling that as a crucial property of the $N = 2$ SYM path-integral measure, there is a dual version of the theory where the fields are the S transformed of the original ones. In particular, the dual effective coupling constant is

$$\tau_D = -\frac{1}{\tau}. \tag{17}$$

Now observe that the $\Gamma(2)$ symmetry can also be interpreted in the following way. Let us schematically denote by I and II the original theory and its dual, respectively,

$$S \cdot I = II. \tag{18}$$

As T^2 is a symmetry of I theory, we have $T^2 \cdot I = I$. On the other hand, the properties of the $N = 2$ measure imply that the asymptotic analysis still holds for the dual

theory II, that is, $T^2 \cdot II = II$. It follows that

$$S^{-1}T^2S \cdot I = S^{-1}T^2 \cdot II = S^{-1} \cdot II = I, \tag{19}$$

implying that besides T^2 also $S^{-1}T^2S$ is in the symmetry group. Repeating the steps in (19) we generate all $\Gamma(2)$, which is the symmetry group of I and of its dual version II. This fact suggests that the higher-order terms in $S[A,\overline{A}]$ should have a structure such that the asymptotic behavior (16) still holds for the dual theory II. In particular, the asymptotic behavior $\tau \sim \frac{2i}{\pi} \ln a$ is reproduced in the dual theory as $\tau_D \sim -\frac{i}{\pi} \ln a_D$. Since these points are in the spectrum it follows by (17) that $\tau = 0$ for some u. This is the $u = \Lambda^2$ point. A similar property should still hold for the higher-order terms in $S[A,\overline{A}]$. Therefore, in order to preserve the relation $S \cdot I = II$, it seems that $\mathcal{K}(a,\overline{a})$ should vanish at the puncture $u = \Lambda^2$. Furthermore, modular invariance of $\mathcal{K}$ together with the relation $u(\tau + 1) = -u(\tau)$ implies that $\mathcal{K}$ is invariant under $u \to -u$. It follows that $\mathcal{K}$ should vanish at $u = \pm\Lambda^2$. This argument may be better formulated by requiring that the T transformation $\tau \to \tau + 1$, which corresponds to $a \to e^{-\pi i/2}a$ (i.e., $u \to -u$), be an invariance of $\mathcal{K}$, namely,

$$\mathcal{K}(e^{-\pi i/2}A, e^{\pi i/2}\overline{A}) = \mathcal{K}(A,\overline{A}). \tag{20}$$

We now show that there is a natural choice for $\mathcal{K}(A,\overline{A})$ which fulfills all the above conditions. Namely, we propose that

$$\mathcal{K}(A,\overline{A}) = \alpha \frac{e^{-\varphi[\mathcal{G}(A),\overline{\mathcal{G}(A)}]}}{|\mathcal{G}^2(A) - \Lambda^4|}, \tag{21}$$

where α will be fixed using the one-instanton calculation in [8]. By (13) it follows that $\mathcal{K}(A,\overline{A})$ can also be written in the form $\mathcal{K}(A,\overline{A}) = 4\alpha\pi^2 e^{2\varphi_{\mathrm{sw}}} |\mathcal{G}^2(A) - \Lambda^4|$ or

$$\mathcal{K}(A,\overline{A}) = 2\alpha\pi e^{\varphi_{\mathrm{sw}}[\mathcal{G}(A),\overline{\mathcal{G}(A)}] - \frac{\varphi}{2}[\mathcal{G}(A),\overline{\mathcal{G}(A)}]}. \tag{22}$$

In the following we will show that the solution (21) has the following properties:

(1) It is modular invariant.

(2) Asymptotically $\mathcal{K}(A,\overline{A}) \sim 2\alpha \ln A/\Lambda \ln \overline{A}/\Lambda$.

(3) The above is the unique singularity of $\mathcal{K}$.

(4) The zeros of $\mathcal{K}$ are precisely at the punctures.

(5) Besides the logarithmic terms the asymptotic expansion of $\mathcal{K}(A,\overline{A})$ contains terms like $(A/\Lambda)^{-4j} (\overline{A}/\Lambda)^{-4k}$ as expected from the instanton contributions (see also [8]).

We now make a few remarks concerning point (1). Strictly speaking, a function $G(A,\overline{A})$ is said to be modular invariant if $G(\gamma \cdot A, \gamma \cdot \overline{A}) = G(A,\overline{A})$, $\gamma \in SL(2,\mathbf{Z})$. The function $\mathcal{K}(A,\overline{A})$ has the invariance $T \circ \mathcal{K}(A,\overline{A}) = \mathcal{K}(A,\overline{A})$ and $S \circ \mathcal{K}(A,\overline{A}) = \mathcal{K}(A,\overline{A})$. Whereas in the first case there is not any change in the functional structure of $\mathcal{K}$, in the case of the S-action one has $S \circ \mathcal{K}(A,\overline{A}) = \mathcal{K}_D(A_D,\overline{A}_D)$ [6]. A similar situation arises in the case of $u = \mathcal{G}(a)$. Since under a $SL(2,\mathbf{C})$ transformation $\mathcal{F}(a)$ and $aa_D/2$ have the same transformation properties, it follows that under $SL(2,\mathbf{C})$ one has $\tilde{\mathcal{G}}(\tilde{a}) = \mathcal{G}(a)$. However, observe that

the T-action $\tau \rightarrow \tilde{\tau} = \tau + 1$ is generated by a phase change of a. In particular, under $a \rightarrow \tilde{a} = e^{\pi i n/2}a$, we have $\mathcal{F}(\tilde{a}) = e^{\pi i n}\mathcal{F}(a) - e^{\pi i n}na^2/2, \tilde{a}_D = \partial_{\tilde{a}}\mathcal{F}(\tilde{a}) = e^{\pi i n/2}a_D - ne^{\pi i n/2}a, \tau \rightarrow \tilde{\tau} = \tau - n$. Concerning the S-action we have $\mathcal{G}_D(a_D) = S \circ \mathcal{G}(a) = \mathcal{G}(a)$. Therefore, since $\mathcal{K}(A,\overline{A})$ in (21) is expressed in terms of $\mathcal{G}(A)$ and $\overline{\mathcal{G}(A)}$, it follows that $\mathcal{K}_D(A_D,\overline{A}_D) = \mathcal{K}(A,\overline{A})$ as it should be [6]. Under the transformation $a \rightarrow \tilde{a} = e^{-\pi i/2}a$, corresponding to the T-action $\tau \rightarrow \tilde{\tau} = \tau + 1$, we have $\mathcal{G}(\tilde{a}) = -\mathcal{G}(a) = -\tilde{\mathcal{G}}(\tilde{a})$. On the other hand, by the $\mathbf{Z}_2$ automorphism of $\mathcal{M}_{\mathrm{SU}(2)}$, we have $e^{\varphi(-u,-\overline{u})} = e^{\varphi(u,\overline{u})}$, so that by (21) $\mathcal{K}$ satisfies Eq. (20). Finally, we observe that by the $\Gamma(2)$ symmetry $\mathcal{G}(\gamma \cdot A) = \mathcal{G}(A)$, we have $\mathcal{K}(\gamma \cdot A, \gamma \cdot \overline{A}) = \mathcal{K}(A,\overline{A})$.

In order to consider point (2) we first write down the asymptotic expansions

$$\mathcal{F} = a^2 \left[\frac{i}{\pi} \ln \frac{a}{\Lambda} + \sum_{k=0}^{\infty} \mathcal{F}_k \left(\frac{a}{\Lambda} \right)^{-4k} \right], \quad (23)$$

$$\tau = \frac{2i}{\pi} \ln \frac{a}{\Lambda} + \frac{3i}{\pi} + \sum_{k=0}^{\infty} \mathcal{F}_k(1 - 4k)(2 - 4k)\left(\frac{a}{\Lambda} \right)^{-4k}. \quad (24)$$

By (1) and (23) it follows that the asymptotic expansion for $u = \mathcal{G}(a)$ is

$$\mathcal{G}(a) = a^2 \sum_{k=0}^{\infty} \mathcal{G}_k \left(\frac{a}{\Lambda} \right)^{-4k}, \quad \mathcal{G}_0 = \frac{1}{2}, \quad (25)$$

where $\mathcal{G}_k = 2\pi i k \mathcal{F}_k$, for $k > 0$. Concerning the instanton contributions $\mathcal{F}_k, k > 0$, these are determined by the recursion relations [3]

$$\mathcal{G}_{n+1} = \frac{1}{8\mathcal{G}_0^2(n+1)^2} \left[(2n-1)(4n-1)\mathcal{G}_n + 2\sum_{j=0}^{n-1} \mathcal{G}_{n-j} \right.$$
$$\left. \times \left(\mathcal{G}_{j+1}\mathcal{G}_0 c(j,n) - \sum_{k=0}^{j+1} \mathcal{G}_{j+1-k}\mathcal{G}_k d(j,k,n) \right) \right], \quad n \geq 0, \quad (26)$$

where $c(j,n) = 2j(n-j-1) + n - 1, d(j,k,n) = [2(n-j) - 1][2n - 3j - 1 + 2k(j - k + 1)]$. To evaluate $\mathcal{F}_0$ we observe that for $u \rightarrow \infty$, Eq. (9) yields $a_D \sim \frac{i}{\pi}\sqrt{2u}(\ln u/\Lambda^2 + \ln 8/e^2)$, whose a term is, by (25), $\frac{i}{\pi}a \ln 4/e^2\Lambda^2$. Comparing with the a term $\frac{i}{\pi}a(\ln e/\Lambda^2 - 2\pi i\mathcal{F}_0)$ in the asymptotic expansion of a_D which follows by (23), we obtain $\mathcal{F}_0 = \frac{i}{2\pi}\ln 4/e^3$.

By (15) and (21), we can rewrite $\mathcal{K}(A,\overline{A})$ in the form

$$\mathcal{K}(A,\overline{A}) = \frac{64\alpha}{\pi^2} \frac{|\mathcal{G}^2(A) - \Lambda^4|[\mathrm{Im}\,\tau(A)]^2}{|A|^4|\hat{\tau}(A) - \tau(A)|^4}. \quad (27)$$

Therefore, by (23)–(25) and (27) we have the asymptotic expansion

$$\mathcal{K}(A,\overline{A}) = \frac{64\alpha}{\pi^2} \left\{ \frac{1}{\pi} \ln \frac{A}{\Lambda}\frac{\overline{A}}{\Lambda} + \frac{3}{\pi} + \frac{1}{2i} \sum_{k=0}^{\infty} \mathcal{F}_k(1 - 4k) \right.$$
$$\left. \times (2 - 4k)\left[\left(\frac{A}{\Lambda} \right)^{-4k} + \left(\frac{\overline{A}}{\Lambda} \right)^{-4k} \right] \right\}^2$$
$$\times \frac{|[\sum_{k=0}^{\infty}\mathcal{G}_k(\frac{A}{\Lambda})^{-4k}]^2 - (\frac{A}{\Lambda})^{-4}|}{|\frac{2i}{\pi} + \sum_{k=0}^{\infty}\mathcal{F}_k 4k(4k-2)(\frac{A}{\Lambda})^{-4k}|^4}. \quad (28)$$

Extracting the leading term we get

$$\mathcal{K}(A,\overline{A}) \sim 2\alpha \ln \frac{A}{\Lambda} \ln \frac{\overline{A}}{\Lambda}, \quad (29)$$

which reproduces the expected behavior.

Let us now consider point (3) above. To show that $e^{-\varphi}/|u^2 - \Lambda^4|$ has no other divergences outside $u = \infty$, it is useful to consider the form

$$\mathcal{K}(a,\overline{a}) = 4\alpha\pi^2|a'|^4|u^2 - \Lambda^4|(\mathrm{Im}\,\tau)^2, \quad (30)$$

and to notice that by (9), $a'(u)$ is logarithmically divergent for $u \rightarrow \pm\Lambda^2$. Therefore, by (30) $\mathcal{K}$ vanishes at $u =$

$\pm\Lambda^2$. It follows that $\mathcal{K}$ is finite everywhere except for the asymptotic divergence (28). Furthermore, since the only zeros of the Poincaré metric e^{φ} are at the puncture at $u = \infty$, where $\varphi \sim -2\ln(|u|\ln|u|)$, whereas the unique divergences come from the punctures at $u = \pm\Lambda^2$, where $\varphi \sim -2\ln(|u \mp \Lambda^2|\ln|u \mp \Lambda^2|)$, it follows that $\mathcal{K}(a,\overline{a})$ has zeros at $a = -i4\Lambda/\pi, a = 4\Lambda/\pi$ [and their $\Gamma(2)$ transformed]. Finally, note that point (5) follows explicitly from the asymptotic expansion (28).

Let us comment on the possible higher-order terms in S. According to [6], besides $\mathcal{K}$, the possible higher-order terms do not seem to be modular invariants. On the other hand, as $u(\gamma \cdot \tau) = u(\tau), \gamma \in \Gamma(2)$, possible noninvariant terms should imply that at the same point of the moduli space of quantum vacua $\mathcal{M}_{\mathrm{SU}(2)}$ there are inequivalent theories. This would break the highly symmetric structure coming from the lower-order part of the action. In this context one should investigate whether the nice observation in [9] that the S duality transformation extends to the full effective action implies that actually the higher-order terms in S are modular invariants. In order to better illustrate this point, we consider the following decomposition $S[A,\overline{A}] = \hat{S}[A,\overline{A}] + \frac{1}{4\pi}\mathrm{Im}[\int d^4x\,d^4\theta\,\mathcal{F}(A)]$. From the definition given in [6], it follows that $\hat{S}[A,\overline{A}]$ is T invariant. This happens also for the T-action on the higher-order part of the dual theory $\hat{S}_D[A_D,\overline{A}_D]$. So that, the fact that $S^2 = \mathbf{I}$ and $(ST)^3 = \mathbf{I}$, indicates that $\hat{S}_D[A_D,\overline{A}_D] = \hat{S}[A,\overline{A}]$. On the other hand, it was argued in [6] that only $\mathcal{K}$ is modular invariant. We consider two possibilities to further investigate this aspect. The first one is that there are not higher-order terms in S besides $\mathcal{K}$. This would imply that

$$S[A,\overline{A}] = 2\alpha\pi \int d^4x\, d^4\theta\, d^4\overline{\theta}\; e^{\varphi_{\rm sw}[\mathcal{G}(A),\overline{\mathcal{G}(A)}] - (\varphi/2)[\mathcal{G}(A),\overline{\mathcal{G}(A)}]} + \frac{1}{4\pi} \,{\rm Im}\!\left(\int d^4x\, d^4\theta\, \mathcal{F}(A)\right). \qquad (31)$$

Another possibility is that the terms coming from the action of $\delta/\delta A_D$ on the dual prepotential, and which break modular invariance (see [6] for details), are actually vanishing. In particular, whereas when all functional derivatives act on the exponentiated dual prepotential there is not any breaking of modular invariance, the other contributions have the effect of modifying the structure of $\hat{S}[A,\overline{A}]$. In other words, forgetting the functional derivative in the right-hand side (rhs) of

$$i^n\, \frac{\delta^n}{\delta A_D^n}\, \exp\!\left[-i\int d^4x\, d^4\theta\, \mathcal{F}_D(A_D)\right]$$
$$= \exp\!\left[-i\int d^4x\, d^4\theta\, \mathcal{F}_D(A_D)\right]\!\left(A + i\,\frac{\delta}{\delta A_D}\right)^{n-1} A$$

would imply that $\hat{S}$ is modular invariant. On the other hand, because of the functional derivative in the rhs, the above expression contains "$\delta(0)$" terms. A possible way to take care of these infinities is to use dimensional regularization where, as is well known, the "$\delta(0)$" terms vanish for dimensional reasons (see, for example, [12]).

Let us now fix the constant α. To this end we use the one-instanton calculation in [8]

$$\mathcal{K}_I(A,\overline{A}) = \frac{1}{32\pi^2}\left(\frac{A}{\Lambda}\right)^{-4} \ln\frac{A}{\Lambda}\frac{\overline{A}}{\Lambda} + {\rm c.c.} \qquad (32)$$

whose coefficient is $1/4$ that in [8]. This is a consequence of the fact that our Λ is $\sqrt{2}$ times the scale considered in [8] where the Pauli-Villars regularization scheme was chosen. By (28) the coefficient of the $(A/\Lambda)^{-4}\ln\frac{A}{\Lambda}\frac{\overline{A}}{\Lambda}$ term in the asymptotic expansion of $\mathcal{K}(A,\overline{A})$ is $2\alpha\{-3\pi i\mathcal{F}_1 + (3 - \pi i\mathcal{F}_0)[2(\mathcal{G}_1 - 1) + 8\pi i\mathcal{F}_1]\} = -\alpha(\frac{9}{4} + \ln 2)$, where we used the fact that $\mathcal{G}_0 = 1/2$, $\mathcal{F}_0 = \frac{i}{2\pi}\ln 4/e^3$, and $\mathcal{G}_1 = 2\pi i\mathcal{F}_1 = 1/4$. Comparing $-\alpha(\frac{9}{4} + \ln 2)$ with the coefficient in (32) we obtain

$$\alpha = -\frac{1}{8(9 + 4\ln 2)\pi^2}. \qquad (33)$$

An interesting point concerning the structure of the logarithmic terms in (28) is that besides $\ln(A/\Lambda)\ln(\overline{A}/\Lambda)$ and $[(A/\Lambda)^{-4} + (\overline{A}/\Lambda)^{-4}]\ln(A\overline{A}/\Lambda^2)$, considered in [7,8], there are the terms $\ln^2 A/\Lambda$ and $\ln^2 \overline{A}/\Lambda$. The fact that these should appear in the asymptotic expansion of $\mathcal{K}$ can also be seen by a simple modular invariance argument. Namely, whereas $(A/\Lambda)^{-4}\ln(A\overline{A}/\Lambda^2)$ is invariant under $A \to e^{-\pi i/2}A$, we have $2\ln(e^{-\pi i/2}A/\Lambda)\ln(e^{\pi i/2}\overline{A}/\Lambda) = 2\ln(A/\Lambda)\ln(\overline{A}/\Lambda) + \pi i(\ln A/\Lambda - \ln \overline{A}/\Lambda)\pi^2/2$, so that in order to satisfy (20) one needs more logarithmic terms. It is easy to see that these should be of the form $(\ln^2 A/\Lambda + \ln^2 \overline{A}/\Lambda)$. We observe that (28) also passes

this test. We also note that once one makes the natural choice of using $({\rm Im}\,\tau)^2$ to reproduce both $\ln\frac{A}{\Lambda}\frac{\overline{A}}{\Lambda}$ and $\ln^2\frac{A}{\Lambda}\frac{\overline{A}}{\Lambda}$, then modular invariance essentially fixes $\mathcal{K}$. In this context it is crucial that $|A|^{-4}\,|\hat{\tau} - \tau|^{-4}$ has the same transformation properties of $({\rm Im}\,\tau)^2$ and, as requested by (16) and (32), cancels the global $|A|^4$ factor in the asymptotic expansion of $|\mathcal{G}^2(A) - \Lambda^4|$ [see (28)].

In conclusion, we observe that similar investigations can be extended to more general cases. For example, in the SU(3) case the uniformization of the quantum moduli space has been considered in [13]. Also, some consequences concerning the Wilsonian renormalization group equation should be further investigated [14,15].

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