Stress Condensation in Crushed Elastic Manifolds

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We discuss an M-dimensional phantom elastic manifold of linear size L crushed into a small sphere of radius $R \ll L$ in N-dimensional space. We investigate the low elastic energy states of 2-sheets ($M = 2$) and 3-sheets ($M = 3$) using analytic methods and lattice simulations. When $N \geq 2M$ the curvature energy is uniformly distributed in the sheet and the strain energy is negligible. But when $N = M + 1$ and $M > 1$, both energies appear to be *condensed* into a network of narrow $M - 1$ dimensional ridges. The ridges appear straight over distances comparable to the confining radius R. [S0031-9007(97)02345-4]

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It has long been known that a thin elastic plate will develop narrow ridges under a variety of compressive boundary conditions. These ridges may be seen daily in the way a pillowcase or trouser leg deforms to wrap its contents, often exhibiting a diamond pattern familiar from compression studies of thin metal cylinders [1]. The linear scars in a crumpled sheet of paper are also a record of this mechanism [2]. Recently, it was discovered that the structure of these ridges could be accounted for using linear elasticity theory, valid in the limit that the ridge length X is much greater than the plate thickness h [3,4]. Witten and Li used a scaling argument to predict that the ridge width $w \simeq h^{1/3}X^{2/3}$ and the total elastic energy $E \simeq Yh^3(X/h)^{1/3}$, where Y is the Young's modulus [3]. Lobkovsky *et al.* verified these scaling laws using both numerical simulations and an asymptotic analysis of the Von Karman equations for a thin plate [4]. His simulations showed that the material strains and curvatures decay rapidly to zero in the direction transverse to the ridge. The length scale of this decay is the ridge width w, which goes to zero with the thickness of the plate. However, these ridges were analyzed only in idealized, symmetrical deformations. Their applicability to stochastically crumpled sheets has not been explicitly shown.

Ridge formation is a mode of spontaneous condensation of energy into a small subset of the available volume. As such, it resembles the spontaneous organization of dislocations into grain boundaries in a strained crystal or the formation of Prandtl boundary layers in laminar fluid flow [5,6].

To understand the necessary conditions for this condensation, and its consequences, it is useful to consider the general problem of an M-dimensional elastic manifold crushed by a hypersphere in N-dimensional space. In Ref. [7] we derived the elastic energy functional of the manifold by considering the small thickness limit of an N-dimensional elastic solid with an extent $O(L)$ in M directions and a thickness h in $N - M$ transverse directions. We found that a deformed hypersurface ($M = N - 1$) under an appropriate boundary condition exhibits a ridge with scaling properties analogous to those of Ref. [4]. In this Letter we explore the more general situation of an M-sheet with a free boundary confined by a small sphere. Using analytic and numerical methods, we demonstrate two complementary behaviors.

Case 1 ($N \geq 2M$): The energy E_c associated with curvature is distributed uniformly in the manifold, and the energy E_s associated with strain is negligible by comparison.

Case 2 ($N = M + 1 > 2$): The strain energy and curvature energy are comparable, with $E_s/E_b \simeq 0.2$. Both energies are condensed into a network of narrow ($M - 1$)-dimensional ridges as described in Refs. [4] and [7].

In the remaining cases, namely, $M + 1 < N < 2M$, we anticipate that the strain energy will also be comparable to the curvature energy. We have not investigated these cases in detail.

The differential geometry of the deformed manifold is conveniently discussed using a Euclidean coordinate patch on the *flat* manifold ($x_\alpha | \alpha \in [1, M]$), called the manifold coordinate system [9]. Any deformation may then be written as a map $\vec{r}(x)$ from the manifold coordinates to N-dimensional Euclidian space. The strain tensor is $u_{\alpha\beta}(x) = (1/2)(\partial_\alpha \vec{r} \cdot \partial_\beta \vec{r} - \delta_{\alpha\beta})$ and the extrinsic curvature tensor is $\vec{K}_{\alpha\beta} = \partial_\alpha \partial_\beta \vec{r}$ [9]. Under the usual assumptions of linear elasticity theory ($u_{\alpha\beta} \ll 1$ and $\partial_\gamma u_{\alpha\beta} \ll |\vec{K}_{\mu\nu}| \ll 1/h$ [10]), the elastic energy functional becomes

$$E = \int dx^M \{\mu(u_{\alpha\beta})^2 + (\lambda/2)(u_{\alpha\alpha})^2 + \kappa \vec{K}_{\alpha\beta} \cdot \vec{K}_{\alpha\beta} + c_0 \kappa \vec{K}_{\alpha\alpha} \cdot \vec{K}_{\beta\beta}\}, \quad (1)$$

where $\mu, \lambda \simeq Yh^{N-M}$ are the Lamé coefficients, $\kappa \simeq Yh^{N-M+2}$ is a bending rigidity, and c_0 is a dimensionless constant [7]. Summation over repeated indices is implied. We refer to the terms quadratic in the strains as the strain energy E_s, and the terms quadratic in the curvatures as the curvature energy E_c [11]. We consider only *phantom* (not self-avoiding) manifolds in this paper, since we don't expect self-avoidance to alter the qualitative conclusions. Our compressive boundary condition is a frictionless spherical shell with an initial radius $R_i > L$. To crush the manifold the radius is slowly decreased to a final value $R \ll L$.

In our simulations we model a thin elastic 2-sheet ($M = 2$) using a triangular network of nodes connected by springs (see Fig. 1), following the work of Seung and Nelson [12] . The discretized strain energy is the sum over the Hooke's law energy of each spring

$$E_s = \sum_i \frac{1}{2} c_s (l_i - l_0)^2, \quad (2)$$

where c_s is a spring constant controlling the strain energy, l_i is the length of spring i, and l_0 is the lattice constant.

To discretize the curvature energy it is convenient to begin by defining the normal tensor for a two-dimensional lattice in N-dimensional Euclidean space. If $\vec{a}(x)$ and $\vec{b}(x)$ are two independent vectors tangent to the 2-sheet at x, then we may define a normal tensor via

$$(\mathsf{n}[a,b])_{i_1 \cdots i_{N-2}} = \epsilon_{i_1 \cdots i_N} a_{i_{N-1}} b_{i_N}, \quad (3)$$

where a_i and b_i are the Cartesian components of $\vec{a}$ and $\vec{b}$, and ϵ is the antisymmetric Levi-Civita tensor. In 3-space this reduces to the usual normal vector $\mathsf{n} = \vec{a} \times \vec{b}$. Defining the inner product

$$\mathsf{n}_1 \cdot \mathsf{n}_2 = \frac{1}{(N-M)!} (\mathsf{n}_1)_{i_1 \cdots i_{N-M}} (\mathsf{n}_2)_{i_1 \cdots i_{N-M}} \quad (4)$$

with $M = 2$, the unit normal tensor may be expressed as $\hat{\mathsf{n}} = \mathsf{n}/\sqrt{\mathsf{n} \cdot \mathsf{n}}$. By substituting $\vec{a} = \partial_1 \vec{r}$ and $\vec{b} = \partial_2 \vec{r}$, it is straightforward to prove that $\partial_\alpha \mathsf{n} \cdot \partial_\alpha \mathsf{n} = \vec{K}_{\alpha\beta} \cdot \vec{K}_{\alpha\beta}$. The curvature energy can therefore be represented using the square of the discrete derivative

$$E_c = \sum_{\langle ij \rangle} \frac{1}{2} c_b |\hat{\mathsf{n}}_i - \hat{\mathsf{n}}_j|^2, \quad (5)$$

where c_b is a bending constant, $\hat{\mathsf{n}}_i$ is the unit normal of triangle i (see Fig. 1), and the sum is taken over each pair of triangles which share a common edge. Comparison with Eq. (1) gives $\mu, \lambda \simeq c_s/l_0^2$, $\kappa \simeq c_b/l_0^2$, and $c_0 = 0$ [12]. The effective thickness of the 2-sheet is determined by the ratio $c_b/c_s \simeq h^2$. We take $(c_b/c_s)^{1/2} \gtrsim l_0/6$ to minimize lattice effects in our simulations.

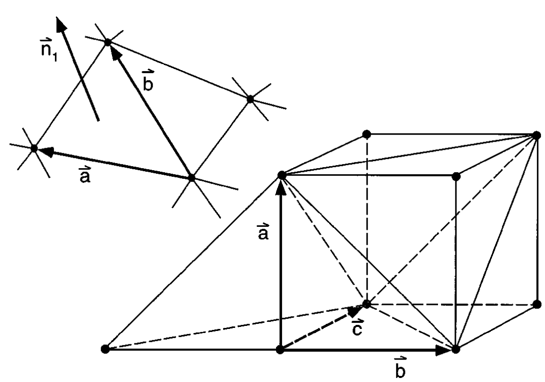

FIG. 1. A portion of the triangular lattice and the simple cubic lattice used to discretize a 2-sheet ($M = 2$) and a 3-sheet ($M = 3$), respectively.

The discrete model of an elastic 3-sheet ($M = 3$) is qualitatively similar. The 3-sheet is a simple cubic lattice of nodes connected by springs. As shown in Fig. 1, there are edge springs with constant c_e and length l_0, and diagonal springs with constant c_d and length $\sqrt{2}\, l_0$. The requirement that the lattice be elastically isotropic for small strains is $c_d = 2c_e$. To calculate the curvature energy we first define the normal tensor on a 3-manifold in N space

$$(\mathsf{n}[a,b,c])_{i_1 \cdots i_{N-3}} = \epsilon_{i_1 \cdots i_N} a_{i_{N-2}} b_{i_{N-1}} c_{i_N}, \quad (6)$$

where $\vec{a}(x)$, $\vec{b}(x)$, and $\vec{c}(x)$ span the tangent space of the manifold at (x). Using the inner product Eq. (4) with $M = 3$, we again have $\partial_\alpha \mathsf{n} \cdot \partial_\alpha \mathsf{n} = \vec{K}_{\alpha\beta} \cdot \vec{K}_{\alpha\beta}$. Each unit cube is conceived as five adjacent tetrahedra with the springs as edges. Each cube then has four "corner" tetrahedra surrounding one "middle" tetrahedron. The sum in Eq. (5) is taken over all pairs of tetrahedra that share a common triangular face (see Fig. 1). Lastly, we relate the bending constant c_{cc} for a pair of corner tetrahedra to the bending constant c_{cm} for a corner-middle pair. Isotropy requires $c_{cm} = 2c_{cc}$.

The initial condition in our simulations is a hexagonal 2-sheet ($M = 2$) or a roughly spherical 3-sheet ($M = 3$) with a small random displacement added to the positions of the nodes [13]. We model a hyperspherical container of radius r_0 with the potential $V_{\text{sphere}} = \sum_i (r_i/r_0)^{12}$, where r_i is the distance from the origin to node i. We use the rms radius of the manifold as our measure of the confining radius R: $R^2 = \sum_i r_i^2 / (\sum_i 1)$. To crush the manifold, the value of r_0 is repeatedly decreased and the energy of the manifold minimized at each step using a conjugate gradient routine [14]. Using this method, a hexagonal 2-sheet with long diameter $L = 160 l_0$ may be crushed to a radius $r_0 = 20 l_0$ in three days of CPU time on an IBM RISC 6000. An approximately spherical ball with diameter $L = 30 l_0$ may be crushed to $r_0 = 6.0 l_0$ in about five days.

The simplest instance of Case 1 is a thin elastic rod crushed within a circle in 2-space. As the radius of the circle is decreased, the rod buckles and develops a curvature of $O(1/R)$. Because the rod is free to reptate parallel to its length, its strain energy remains identically zero. Analogously, a thin plate can curl without strain into an arbitrarily small sphere in 4-space. One possible embedding which demonstrates this is $\vec{r}^{\,\star}(x) = \rho[\cos(x_1/\rho), \sin(x_1/\rho), \cos(x_2/\rho), \sin(x_2/\rho)]$, where $\rho = R/\sqrt{2}$. Note that $|\vec{r}^{\,\star}|^2 = R^2$ everywhere. One can verify by substitution that the strain tensor $u_{\alpha\beta}$ is identically zero, and the curvatures satisfy $|\vec{K}_{11}(x)| = |\vec{K}_{22}(x)| = \sqrt{2}/R$, $\vec{K}_{12}(x) = 0$. Thus $E_{\text{tot}} \simeq \kappa (L/R)^2$. It is interesting to note that the rotational symmetry of the manifold is globally broken by the embedding $\vec{r}^\star$. Calculation of the normal correlation function [Eqs. (3) and (4) are valid in the continuum limit] gives $\hat{n}(x) \cdot \hat{n}(x + x') = \cos(x_1'/\rho)\cos(x_2'/\rho)$.

The key feature of the embedding $\vec{r}^\star$ is the separation of the manifold coordinates into independent, two-dimensional subspaces of 4-space. Analogously, whenever $N \geq 2M$ we may choose the embedding

$$\vec{r}^\star = \rho[\cos(x_1/\rho), \sin(x_1/\rho), \cos(x_2/\rho), \sin(x_2/\rho), \ldots, \cos(x_M/\rho), \sin(x_M/\rho), 0, 0, \ldots], \quad (7)$$

where $\rho = R/\sqrt{M}$. This embedding is an existence proof that the manifold need not strain during the compression. It is also the *global* minimum of the elastic energy at fixed R (neglecting corrections within R of the manifold boundary). As was true for the sheet, the embedding $\vec{r}^\star(x)$ breaks the rotational symmetry of the manifold for all $M > 1$. With this broken symmetry comes a degeneracy. Each rotation of the manifold coordinate system $x \longrightarrow y$ gives a distinct minimum-energy configuration $\vec{r}^\star(y)$.

Our simulations of a 2-sheet in 4-space and a 3-sheet in 6-space behave as anticipated for Case 1. In both instances we observe the minimum energy embedding discussed above, with a curvature energy density uniform to 10% and negligible strain energy $E_s \lesssim 10^{-3}E_c$. The only significant deviation from the ideal embedding is that the manifold flattens out within R of the manifold edge. As a result, the curvature energy density decays to zero near the edge and there is a systematic downward correction to the total energy of the form $E_{\text{tot}} \sim [L - (\text{const})R]^M/R^2$. Depending upon the initial condition, we also observe several metastable states with a nonuniform curvature energy density and a higher total energy. As compression proceeds, these metastable states make the transition to the minimum energy embedding.

Our understanding of Case 2 deformations is incomplete. It is based on previous analytical and numerical work for a variety of simple deformations, including some on general hypersurfaces [4,7]. These cases all show the condensation of the elastic energy into a fraction $O(h/X)^{1/3}$ of the manifold volume, suggesting that a generic compression yields similar condensation. The simple cases also exhibit a strict proportionality between strain and curvature energy: $\lim_{(h/X)\to 0} E_s = E_c/5$.

We simulated Case 2 deformations using a 2-sheet in 3-space and a 3-sheet in 4-space. For the 2-sheet we see the anticipated spontaneous formation of a network of linear ridges. Figure 2 shows the curvature energy density in the manifold coordinates of a hexagonal 2-sheet with long diameter $L = 160l_0$ crushed to a radius $R = 27l_0$. About 40% of the total curvature energy occupies just 2% of the total area. This is due to the formation of point vertices in the 2-sheet. These vertices are the tips of conelike deformations. The next 40% occupies 20% of the area. We see that this energy is condensed into a network of narrow ridges which connect the vertices. The strain energy density is similarly localized. The ratio $E_s/E_c \approx 10\%$. This is consistent with the deviations from the asymptotic value $E_s/E_c = 1/5$ reported by Lobkovsky for short ridges $X \approx 10h$ [4].

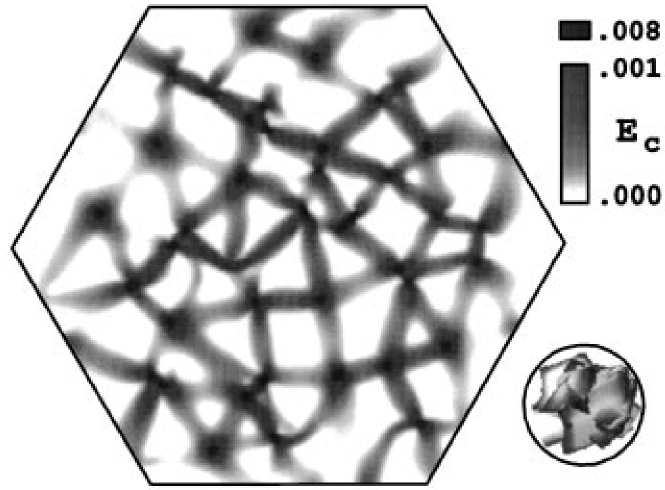

FIG. 2. Curvature energy distribution in a hexagonal 2-sheet with moduli $c_s = 1.0$ and $c_b = 0.025$, and long diameter $L = 160l_0$ crushed to a radius $r_0 = 27l_0$. Darker regions have higher energy density. The crumpled configuration is shown in the lower right.

Figure 2 suggests that the ridges forming the network are of comparable length. This length is similar to the diameter of the confining sphere shown in the lower right corner. If we make the scaling hypothesis that the mean ridge length $\overline{X} \simeq R$, then the number of ridges in the 2-sheet scales as $(L/R)^2$. Since the energy of a single ridge scales as $Yh^3(R/h)^{1/3}$ [4], the total elastic energy E_{tot} should obey $E_{\text{tot}} \simeq YhL^2(h/R)^{5/3}$. Our simulation results are consistent with $E_{\text{tot}} \sim R^{-5/3}$ [15]. However, due to boundary effects and the small values of the aspect ratio $L/h \lesssim 300$, the evidence for this scaling is only suggestive.

Our simulations of an elastic 3-sheet in 4-space attained aspect ratios of only $L/h \simeq 30$. Since the ridge properties derived in Ref. [7] are well defined only in the limit that the ridge is much longer than h, these simulations provide only qualitative support for Case 2. The simulations show the anticipated concentration of strains and curvatures into linear and planar structures. Figure 3 shows two curvature energy isosurfaces in the manifold coordinates of a 3-sheet with diameter $L = 30l_0$ crushed to $r_0 = 12l_0$. The upper isosurface encloses 65% of the total curvature energy and 16% of the volume. We see distinct planar structures. The lower isosurface encloses 23% of the curvature energy and only 3% of the volume. The energy is concentrated on the set of lines where planes intersect. The lines are the analogs of the pointlike vertices in Fig. 2. The ratio E_s/E_c is 15%.

We anticipate scaling behavior for the energy of a crushed hypersurface analogous to that

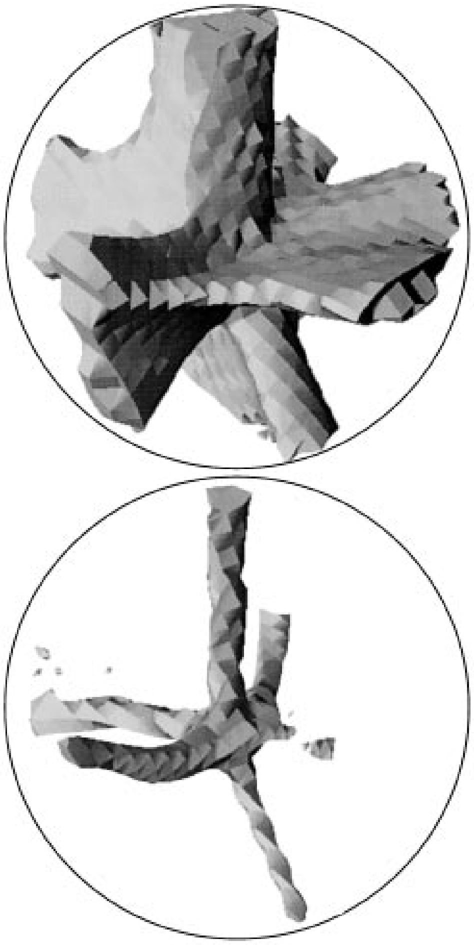

FIG. 3. Curvature energy isosurfaces in the manifold coordinates of an elastic 3-sheet ($L = 30 l_0$) in 4-space crushed to a radius $r_0 = 13 l_0$. The top (bottom) surface encloses 65% (23%) of the total energy in 16% (3%) of the volume.

for a crushed 2-sheet. If the linear size of the ridges is roughly the diameter of the confining sphere, we may generalize the argument for the 2-sheet to infer $E_{\rm tot} \simeq YhL^M(h/R)^{5/3}$ [7]. The limitations of our simulation prevented us from testing this prediction.

This work has implications for real crumpled sheets. It is the first exploration of the distribution of energy in these sheets, and it offers qualitative support for the model of a crumpled sheet as a network of ridges [4]. The possibility of crushing without strain in high dimensions may be relevant for studies of thermal crumpling in general dimensions [16]. More broadly, this work identifies a new mechanism of energy condensation into a small subspace of an available volume. This condensation happens in arbitrary spatial dimensions in one of the simplest organizations of matter—an elastic manifold. Increasingly, the behavior of simple manifolds in general spatial dimensions is invoked to account for fundamental processes [17]. The symmetry-breaking and ridge-forming mechanisms explored here may prove relevant for understanding such behavior.

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