Noise Filtering in Communication with Chaos

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A method, based on fundamental properties of chaotic dynamics, is devised for filtering *in-band* noise of an incoming signal generated by a chaotic oscillator. Initially the $2x \bmod 1$ map is used to illustrate the procedure and then the method is applied to recover the message encoded in a realistic chaotic signal, after the transmitted signal has been contaminated with noise. [S0031-9007(97)02452-6]

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The realization that chaos can be controlled by using small perturbations [1] has opened even more the already wide spectrum of potential applications for chaotic dynamical systems. In particular, aiming at using chaos in communication, small perturbations have been used to make the symbolic dynamics of a chaotic system follow a desired symbol sequence [2]. This permits any message to be encoded in the signal generated by a chaotic oscillator. In addition, it was experimentally demonstrated [3] that it is indeed feasible to use a nonlinear chaotic oscillator as the source of a digital communication signal and that any message can be encoded and then transmitted using chaotic oscillations. As a proof of principle, the double-scroll oscillator [4] was used as the experimental device due to its simplicity and well-known dynamics. A different approach for using chaos in communication has been reported by several other authors [5].

A signal transmitted over a communication line, or channel, is generally subject to noise disturbances. Since noise is typically broadband, one part of its power spectrum is within the frequency range of the transmitted signal (*in-band* noise), while the other part is outside that range (*out-of-band* noise). The in-band noise poses a greater deal of difficulty than the out-of-band noise when the task is to increase the ratio of signal-to-noise power density by lowering the noise power [6]. In this Letter we describe how fundamental properties of chaotic dynamical systems can be used to effectively reduce the in-band noise after the message bearing chaotic signal has been degraded by noise. Our main purpose is to show that a chaotic (noisy) signal carrying a predesigned encoded message can be filtered yielding a less noisy signal. Our practical goal is to encode a message as was shown in Ref. [2], let this message be corrupted by noise during the transmission, and then reproduce a less noisy version of the incoming chaotic signal. This less noisy version is, in a measurable sense, closer to the original signal before it became contaminated with noise. The dynamical basis for our ideas is that the component of the noise in the stable direction is reduced as one propagates forward along the chaotic signal, while the component of the noise in the unstable direction is reduced as one propagates backward. In other words, in the expanding direction the chaotic signal is sensitive to perturbations, whereas in the contracting direction it damps them out. We make use of this contractiveness to filter the in-band noise and, in order to implement these ideas, we employ a method provided by chaos-control-based synchronization, as developed in Ref. [7], to zero-in on the original signal. In this work, we demonstrate how these fundamental ideas from chaotic dynamics can be used to filter in-band noise in a simple nontrivial way, with potential relevant technological applications [8].

Suppose that we have a *transmitted* signal $\{x_n\}_{n\in J}$ (J being the set of non-negative integers), generated by a chaotic oscillator, to which noise $\{\xi_n\}_{n\in J}$ is added along the transmission process. At the receiver, the *incoming* signal is $\{\tilde{x}_n \mid \tilde{x}_n = x_n + \xi_n\}_{n\in J}$. We want to obtain a signal $\{\hat{x}_n\}_{n\in J}$ that has less noise than the incoming signal $\{\tilde{x}_n\}_{n\in J}$. As a demonstration of principle and to show how our ideas on noise filtering work, we first use the Bernoulli shift $2x \bmod 1$ map to generate the signal sequence to be transmitted. Gaussian noise is added to it and, for the purpose of illustration, it is assumed that, at the filter, the equations of motion (in this case the map) that generated the signal are known. This is reasonable since the implementations of the transmitter and of the filter are based on the same dynamics. We show that our filtering method lets the $2x \bmod 1$ signal pass through without attenuation, whereas the noise is attenuated. We also estimate by how much the filter attenuates noise.

To encode a symbol sequence containing a message, we label the left-hand side ($0 \le x < \frac{1}{2}$) of the unit interval with the symbol "0," and the right-hand side ($\frac{1}{2} \le x \le 1$) with the symbol "1." All symbol sequences are admissible for this system and trajectories with almost all initial conditions with respect to Lebesgue measure generate

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symbol sequences in which the binary symbols "0" and "1" have equal likelihood. Using a binary fraction to represent symbol sequences, each symbol sequence is mapped into the real number $r = \sum_{n=1}^{\infty} b_n 2^{-n}$ (referred to as the symbolic state of the system), where b_n is the nth binary symbol generated by the dynamics. We want a specific symbol sequence, one carrying a message, say, the word "chaos." Using the seven-bit ASCII character set, the decimal binary sequence $r_{\rm ch} = 0.0110001101101000$ represents the first two letters "ch." The symbol "0" has been added to initialize each seven-bit character and so the whole word "chaos" is represented by a sequence of 40 symbols. For the particular case of the characters "ch," its decimal binary sequence is the finite precision number 0.388 306. The next iterate of the $2x$ mod 1 map is 0.766 611, which is equivalent to "left shifting" the decimal binary sequence to 0.1100011011010000 and deleting the bit to the left of the decimal point. The first bit to the right of the decimal point (most significant bit) defines whether $r \geq \frac{1}{2}$ or $r < \frac{1}{2}$. It is possible to control the sequence of points at the transmitter in order to make r to be above or below the partition value $r = \frac{1}{2}$ in a desired sequence, an operation that can be executed with trajectory perturbations [2,3,9]. Any desired binary sequence can be produced, meaning that any desired message can be encoded. In Fig. 1 the word "chaos" is represented by the squares as it was encoded by the transmitter, and as this message is transmitted through the channel it becomes corrupted by noise, here introduced as additive 8.5% normally distributed random deviates. The noisy signal is represented in Fig. 1 by circles, and the noise disturbances eventually can kick a point (located closest to $r = 0$ and $r = 1$) out of the unit interval. This difficulty is overcome by extending the interval to the real values

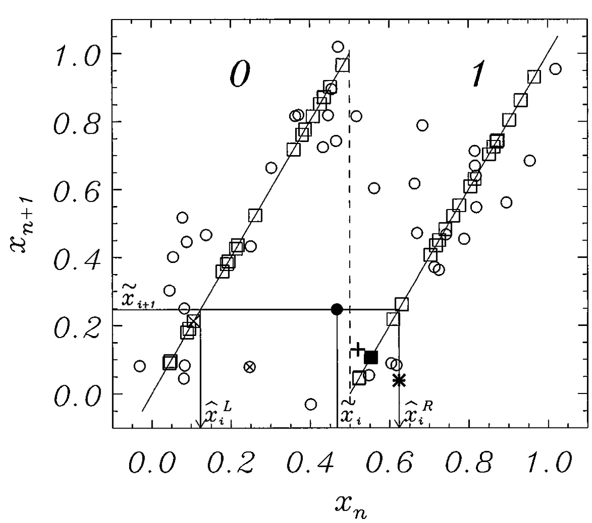

FIG. 1. The Bernoulli shift $2x$ mod 1 map with the encoded word "chaos" to be transmitted (squares), and the same signal with additive Gaussian noise (circles). The point denoted by the filled square on the "1" side of the map is kicked to the "0" side (filled circle) of the map because of noise. $\hat{x}_i^L$ and $\hat{x}_i^R$ correspond to the two preimages of $\tilde{x}_{i+1}$.

(negative values below $r = 0$, and positive values above $r = 1$). The noise disturbances can also kick a point over the partition, in which case the wrong symbol will be assigned at the receiver. The points that are most vulnerable to this difficulty are those located nearest to $r = \frac{1}{2}$. In this case we adopt the following approach. Consider, for instance, the point denoted in Fig 1 by the filled square corresponding to the third bit to the right of the decimal point. This point ($x_i = 0.553\,238 > \frac{1}{2}$, $x_{i+1} = 0.106\,491$) corresponds to the symbol "1." With additive Gaussian noise it becomes ($\tilde{x}_i = 0.467\,611 < \frac{1}{2}$, $\tilde{x}_{i+1} = 0.247\,353$), which corresponds to the symbol "0," in Fig. 1 denoted by the filled circle. Similar problem happens with three other points in the 40-bit sequence. The overall consequence is that at the receiver, without noise reduction, three symbols will be switched and the word "Gbaob" will be received instead of "chaos." To recover the signal (and hence the word "chaos" to be read correctly at the receiver), we need to filter the noise. This is done by repeatedly performing our filtering procedure, which, for each application of the filter, consists of backward iterating each point of the message. Therefore one picks, for instance, the point mentioned above (filled circle in Fig. 1) with coordinates $(\tilde{x}_i, \tilde{x}_{i+1})$. To obtain a less noisy estimate of this point, we backward iterate it. Each backward iteration makes use of the contractiveness of the dynamics along the unstable direction. However, notice that the map is noninvertible, and therefore the backward iteration for this process is not so trivial. For one iterate, there are two preimages for each point, and for the point we chose as example, the two preimages are $\hat{x}_i^L$ and $\hat{x}_i^R$, where L and R stand for "left" and "right," respectively. The preimage to be picked is the one located closest to $\tilde{x}_i$. For this example, the two preimages of the ordinate $\tilde{x}_{i+1}$ are $\hat{x}_i^L = \frac{\tilde{x}_{i+1}}{2}$ and $\hat{x}_i^R = \frac{\tilde{x}_{i+1}}{2} + \frac{1}{2}$. We compare $\tilde{x}_i$ to $\frac{\tilde{x}_{i+1}}{2} + \frac{1}{4}$ (halfway between the two preimages). If $\tilde{x}_i$ is greater than $\frac{\tilde{x}_{i+1}}{2} + \frac{1}{4}$, we pick $\frac{\tilde{x}_{i+1}}{2} + \frac{1}{2}$; otherwise, we pick $\frac{\tilde{x}_{i+1}}{2}$. Here we pick $\hat{x}_i^R = \frac{\tilde{x}_{i+1}}{2} + \frac{1}{2} = 0.623\,676$ as the filtered abscissa of the point. The filtered ordinate (which is the same as the filtered abscissa of the next point) is obtained by following the same steps now considering the ordinate of the next point ($\tilde{x}_{i+1} = 0.247\,335, \tilde{x}_{i+2} = 0.078\,351$), represented in Fig. 1 by the circle with a "$\times$" (the corresponding noiseless point is represented by the square with a "$\times$"). After backward iteration of $\{\tilde{x}_{i+2}\}$ and decision for preimage picking we obtain $\hat{x}_{i+1} = 0.039\,176$. The filtered point is represented in Fig. 1 by the "$*$" mark, already on the right-hand side of the map where it belongs. For the next filtered point the procedure is repeated, and all the way to the last point of the message, completing in this manner the first iterate of the filter. In practice, it can be easily implemented experimentally and quickly executed in a circuit using discrete-analog electronics. In the second iterate, we start with the first filtered point and go all the way through to the last using the same procedure. For the same point we are working with, its position after this second iteration is represented in Fig. 1 by the "+" mark, and it is visible that each iterate of the filter brings the point closer to its original position (filled square) before noise was added. After just four iterations of the filter, the ratio of signal-to-noise power density goes up by a factor of 270. This, more than 2 orders of magnitude increase in the ratio of signal-to-noise power density, is due to the zeroing in effect of the inverse process. Since the slope of the map is 2, the filter shrinks noise by a factor of 2 at each filter iteration. The idea is that if there is a point on the trajectory within a distance ϵ of the true $2x \bmod 1$ point, its preimage lies within a distance $\epsilon/2$. Repeating the procedure for the same point on the next filtering iterate one gets another factor of 2. The noise standard deviation is reduced by a factor of 2 in each preiterate. The signal, however, obeying the dynamics passes unaltered through the filter, with unit gain. The ratio of signal-to-noise power density is multiplied by a factor of 4 in each iterate of the filter.

So, by using the discrete-time $2x \bmod 1$ map, we have demonstrated how our ideas on noise filtering can be implemented. We now apply our filtering procedure to a more realistic and more significant model than the $2x \bmod 1$ map, namely, the continuous-time signals produced by the Lorenz system [10] (with the standard parameter values $\sigma = 10$, $b = 8/3$, and $R = 28$). Its attractor is basically a two-lobe structure, and, if we associate each cycle of an attractor lobe with a binary symbol, then it becomes a source of almost equally likely symbol sequences. For instance, by labeling one lobe with the symbol "0" and the other with the symbol "1," each time the system cycles on the "0" lobe it produces a "0," and likewise for the "1" lobe. The sequence of cycles on the lobes can be controlled by using a simple control technique [11] to generate a desired binary sequence. In doing so we have encoded the message "A mind stretched by a new idea never shrinks back to its former dimensions" [12] in the Lorenz $x(t)$ component. As this signal is transmitted, it becomes infected with noise and at the receiver, time delay embedding is used to reconstruct the attractor, shown in Fig. 2. The return map is then built by Poincaré sampling this attractor with the two branches of the crossing surfaces defined in same way as in Ref. [11]. Since there is no closed algebraic formula for this Poincaré map, we produce a piecewise linear fitting of it, preferable from the point of view of possible technological application, as opposed to a polynomial fitting. And, as we now have the noisy return map and a piecewise linear fitting of it, backward iterations are applied to each point in the same way as was done for the shift map [13]. Figure 3 shows a sequence of return maps, starting with the clean map, followed by the noisy map, then after 2 iterates and after 5 iterates of the filter, as indicated. The power spectrum density of the clean $x(t)$ signal is shown in Fig. 4, curve denoted by the letter a. Curve b represents the power spectrum density of the in-band white Gaussian noise [14] that was added to

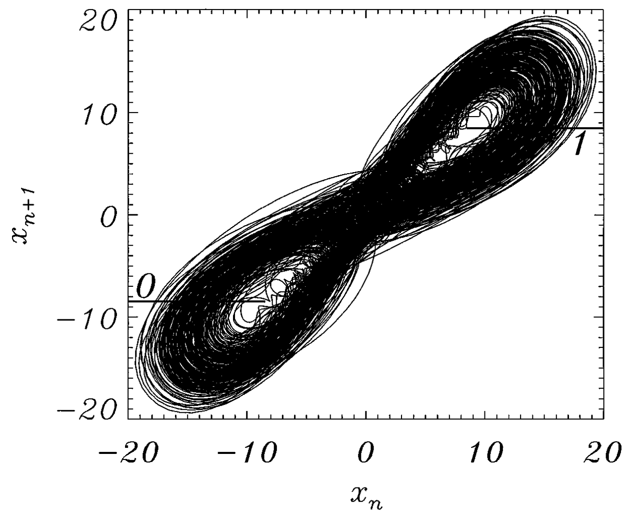

FIG. 2. Lorenz attractor built by applying time delay embedding to the $x(t)$ noisy signal. The two lines intersecting the lobes represent two branches of a Poincaré surface of section to which the binary symbols "0" and "1" have been associated.

the $x(t)$ clean signal. After five iterations of the filter, the ratio of signal-to-noise power density is raised by a factor of 680. The power spectrum of the residual noise is depicted in Fig. 4 by curve c.

The present method's convergence depends on the noise statistics and on the system dynamics. For example, for the shift map it is possible to control the system on a subset of the natural invariant set that avoids the neighborhood of the point $x_n = \frac{1}{2}$. If the noise density is uniform and not too big, there will never be errors and convergence is guaranteed to the correct signal. More general cases depend on the noise density function and on the dynamics of the transmitter. Also, this method can, in principle, be extended to higher-dimensional systems. In that case both backward and forward iterations are necessary:

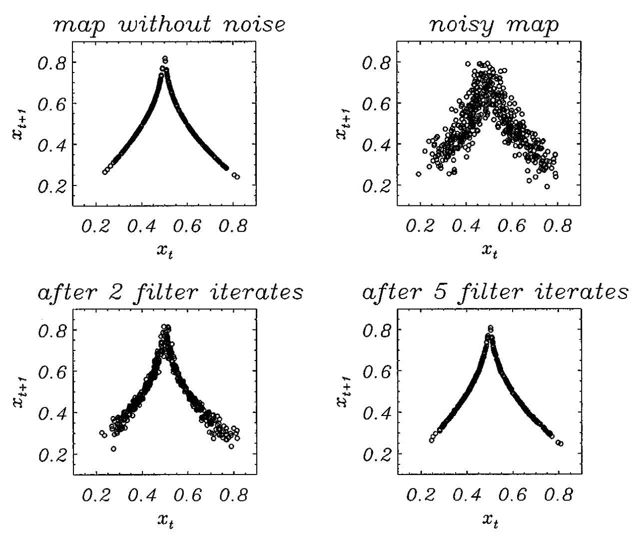

FIG. 3. Return maps built from the clean signal, from the noisy signal, from the signal after 2 iterates of the filter, and after 5 iterates, as indicated.

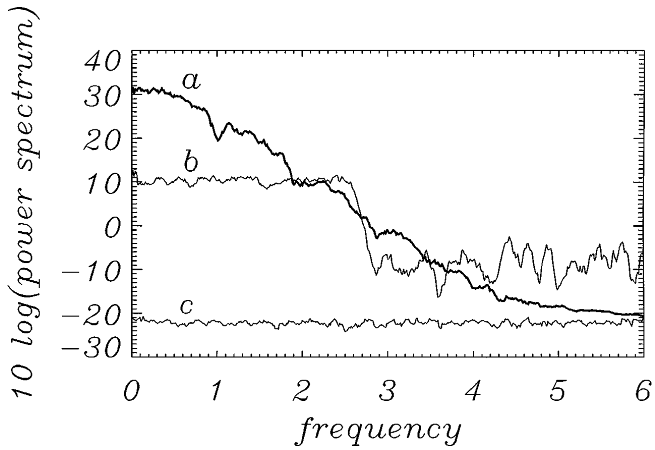

FIG. 4. Power spectrum density of the Lorenz signal prior to noise addition (curve *a*), of the white Gaussian in-band noise (curve *b*), and of the noise residue (curve *c*) after five 5 iterates of the filter.

backward along the unstable direction, forward along the stable direction.

In conclusion, this single input filter has no equivalent in traditional filtering theory [15]. It takes advantage of the nonlinear character of the signal and removes noise in a way that cannot be done with conventional digital signals. For the purpose of practical technological applications, it seems that placing several single input filters in series (cascade form) is desirable. In fact, if the channel is too noisy, it is indispensable to clean the signal periodically along the transmission process.

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