Quarkonia in Hamiltonian Light-Front QCD

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A constituent parton picture of hadrons with logarithmic confinement naturally arises in weak coupling light-front QCD. Confinement provides a mass gap that allows the constituent picture to emerge. The effective renormalized Hamiltonian is computed to $\mathcal{O}(g^2)$, and used to study charmonium and bottomonium. Radial and angular excitations can be used to fix the coupling α, the quark mass M, and the cutoff Λ. The resultant hyperfine structure is very close to experiment. [S0031-9007(97)02413-7]

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The solution of Quantum Chromodynamics in the nonperturbative domain remains one of the most important and interesting unsolved problems in physics. The basic assumption upon which our work is based is that it is possible to *derive a constituent picture for hadrons from QCD* [1–3]. If this is possible, nonperturbative bound state problems in QCD can be approximated as coupled, few-body Schrödinger equations.

To arrive at a constituent approximation to QCD, we first want to separate vacuum fluctuations. This is achieved by formulating the theory on the light-front where the vacuum is trivial in the presence of cutoffs. We start with the canonical light-front Hamiltonian, regulated by a cutoff on light-front energy, and use the similarity renormalization group to renormalize [4]. We expect the Hamiltonian to contain novel finite counterterms when there is spontaneous symmetry breaking [1], but the vacuum is unchanged by spontaneous breaking.

Renormalization of light-front Hamiltonians is more complicated than that of Lagrangians because many symmetries are not kinematically manifest, because there is a separate power counting for longitudinal and transverse directions, and because locality is violated in the longitudinal direction.

The basic idea of the similarity renormalization group is illustrated in Fig. 1. Figure 1(a) schematically shows a regulated bare light-front Hamiltonian matrix. The regulator used for illustration is a cutoff restricting light-front energies, and it makes the Hamiltonian matrix finite. There are different ways to impose the cutoff, but the specifics are not important for this discussion. The bare Hamiltonian contains couplings between all energy scales. This is the source of ultraviolet divergences. In order to renormalize the Hamiltonian, one needs to find counterterms that remove dependence on the cutoff. The similarity renormalization group is based on the following observation: If a Hamiltonian has a band diagonal form, as the Hamiltonian in Fig. 1(b), then no ultraviolet divergence can appear at any finite order of perturbation theory because of the finite width of the Hamiltonian, as long as its matrix elements are finite. Therefore, if one can find a similarity transformation that transforms a bare Hamiltonian H_B as in Fig. 1(a) to a band diagonal Hamiltonian H' as in Fig. 1(b),

$$H' = SH_BS^\dagger + \text{counterterms}, \tag{1}$$

it is possible to identify counterterms by requiring that the matrix elements of the band diagonal Hamiltonian are independent of the regulator. This ensures that physical observables will also be independent of the regulator.

The similarity renormalization can be done in steps or continuously [4]. In what follows, we use a discrete perturbative formulation around the free light-front Hamiltonian.

(a)
$$\begin{pmatrix} a_0 & a & a & \ldots & a & a \\ a & a & a & \ldots & a & a \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ a & a & a & \ldots & a & a \\ a & a & a & \ldots & a & a_{\Lambda_0} \end{pmatrix}$$

(b)
$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{A} & 0 & \ldots & 0 & 0 \\ \mathbf{A} & \mathbf{A} & \mathbf{A} & \ldots & 0 & 0 \\ 0 & \mathbf{A} & \mathbf{A} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{A} & \mathbf{A} \\ 0 & 0 & 0 & \ldots & \mathbf{A} & \mathbf{A}_{\Lambda_0} \end{pmatrix}$$

FIG. 1. (a) An example of a bare regulated Hamiltonian in light-front energy space. Energies run from zero up to an initial cutoff which is indicated by the subscripts of the first and the last diagonal elements. a in this figure schematically indicates nonzero matrix elements, but the matrix elements are not necessarily equal. The Hamiltonian couples states of all energy scales. (b) An example of a band diagonal Hamiltonian. The energies run up to the initial cutoff, but the Hamiltonian couples only states which are close in energy.

With the divergences removed, the remaining task is to adjust finite parts of the counterterms. This is, in principle, achieved by restoring Lorentz invariance and other exact symmetries in physical observables. However, if the similarity transformation can be done analytically, as in the calculation presented here, it is straightforward to use coupling coherence [5], which uniquely fixes all counterterms without explicit reference to underlying symmetries. The basic idea of coupling coherence is the following: In the Hamiltonian restricted by symmetries, albeit not manifest, the strengths of all operators are not independent but depend only on a finite number of independent canonical parameters. Under a full renormalization group transformation (including change of scale and rescaling), the Hamiltonian reproduces itself in form exactly, apart from the change of the explicit cutoff and the running of those few independent couplings. All dependence on the cutoff is absorbed into the independent running couplings. Once one obtains a Hamiltonian that reproduces itself as the cutoff is lowered and subtracts the divergences, any initial cutoff can be sent to infinity.

The coupling coherent solution at second order in a generic interaction v (around the free light-front Hamiltonian h_0) is [2]

$$H_{ab} = \langle a|h_0 + v|b\rangle + \sum_k v_{ak} v_{kb} \left[\frac{\theta(|\Delta_{ak}| - \lambda^2/\mathcal{P}^+)\theta(|\Delta_{ak}| - |\Delta_{bk}|)}{\Delta_{ak}} + \frac{\theta(|\Delta_{bk}| - \lambda^2/\mathcal{P}^+)\theta(|\Delta_{bk}| - |\Delta_{ak}|)}{\Delta_{bk}} \right], \quad (2)$$

where $\Delta ij = E_{0i} - E_{0j}$ is the difference in light-front free energies, $\lambda^2/\mathcal{P}^+$ is the similarity scale, and the sum over states k is limited by the initial cutoff and the explicit similarity cutoff in Eq. (2). A self-energy counterterm is also needed but is not shown (see Ref. [2]).

We would like to note that the similarity renormalization scheme which results in band diagonal effective Hamiltonians is a renormalization method that brings us closer to the desired constituent picture of hadrons. Indeed, if the constituents are massive, then at any finite order of perturbation theory with the band diagonal Hamiltonian, there is only a finite number of Fock states that couple to the lowest Fock component in the hadron. In our work low energy gluons acquire a mass gap due to a mass counterterm and due to the confining interaction, and high multiplicities of gluons are suppressed.

After renormalization is completed, one is left with the effective Hamiltonian band diagonal with the width of a hadronic scale. The effective Hamiltonian contains complicated potentials, which result from eliminating the coupling between high and low energy states. It still contains emission and absorption interactions, but these no longer mix states of high and low energies. However, if one tries to diagonalize the effective Hamiltonian directly, the wave function of a hadron must contain arbitrarily many parton components. Instead, we divide the effective Hamiltonian into a part H_0 which is solved nonperturbatively, and the remaining part V is treated in bound-state perturbation theory. The division is arbitrary, but a choice of H_0 missing an important part of physics of the system under consideration would lead to divergent bound state perturbation theory. Therefore, we want to choose H_0 that approximates the physics relevant for hadronic bound states as closely as possible, and at the same time we want it to be manageable. We take a hint from the constituent quark model and include constituent masses and two-body potentials produced by the similarity transformation.

A major simplification is achieved by not including emission and absorption of low energy gluons in H_0. Once the particle-number-changing interactions are put in V, different Fock states decouple to leading order. The Hamiltonian H_0 provides an approximate $q\overline{q}$ valence quark description of mesons. The errors in approximation can be determined from bound-state perturbation theory in V which links the $q\overline{q}$ to multibody Fock states. Mixing of different Fock components first enters at second-order bound-state perturbation theory. The valence approximation is best justified for heavy quarkonia.

Based on the success of the constituent quark model, it is reasonable to choose a nonrelativistic limit of the effective Hamiltonian for H_0. This approximation, too, is best justified for heavy quarkonia.

We have already used the effective Hamiltonian approach to study properties of heavy-light mesons, in particular B mesons. In this Letter we present numerical results obtained by applying the approach to charmonium and bottomonium for which the approximations are better justified. We fit $1S$, $1P$, and $2S$ levels for both systems. We then predict hyperfine splitting in the charmonium ground state. The prediction is in good agreement with experiment.

We find the effective Hamiltonian to $\mathcal{O}(g_\Lambda^2)$. The effective Hamiltonian, which is generated by the similarity transformation and coupling coherence to order g^2, is band-diagonal in light-front energy with respect to a hadronic scale $\Lambda^2/\mathcal{P}^+$, and it can be written as [3]

$$H_{\text{eff}} = H_{\text{free}} + v_1 + v_2 + v_{2\,\text{eff}}, \quad (3)$$

where H_{free} is the light-front kinetic energy [we remind the reader that the light-front kinetic energy of a particle with transverse momentum $p^\perp$ and longitudinal momentum p^+ is $(p^{\perp 2} + m^2)/p^+$], v_1 is $\mathcal{O}(g)$ emission and absorption with nonzero matrix elements only between states with energy difference smaller than the hadronic scale $\Lambda^2/\mathcal{P}^+$.

Let p_i, k_i be the light-front three-momenta carried by a quark and an antiquark; σ_i, λ_i are their light-front helicities; $u(p,\sigma)$, $v(k,\lambda)$ are their spinors; index $i = 1, 2$ refers to the initial and final states, respectively. Let $\vec{q} = \vec{p}_1 - \vec{p}_2$ be the exchanged momentum and $q^- = q^{\perp 2}/q^+$. v_2 in Eq. (2) is an $\mathcal{O}(g^2)$ instantaneous interaction with the following matrix element for free states containing a

quark and an antiquark:

$$-g_\Lambda^2 \overline{u}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\overline{v}(k_2,\lambda_2)\gamma^\nu v(k_1,\lambda_1)\langle T_a T_b\rangle \frac{1}{q^{+2}}\eta_\mu\eta_\nu\theta\left(\tfrac{\Lambda^2}{\mathcal{P}^+} - |(p_1^- + k_1^-) - (p_2^- + k_2^-)|\right), \quad (4)$$

where $\eta_\mu v^\mu = v^+$ defines the unit vector η_μ. $v_{2\text{eff}}$ includes the $\mathcal{O}(g^2)$ effective interactions generated by the similarity transformation. The effective interactions generated to this order contain one-body and two-body operators. In particular, the effective one-body quark operator is:

$$\frac{\alpha_\Lambda C_F}{2\pi P^+}\left\{2\frac{P^+}{\mathcal{P}^+}\Lambda^2\ln\left(\frac{P^+}{\epsilon\mathcal{P}^+}\right) + 2\frac{P^+}{\mathcal{P}^+}\Lambda^2\ln\frac{x_a^2\frac{P^+}{\mathcal{P}^+}\Lambda^2}{x_a\frac{P^+}{\mathcal{P}^+}\Lambda^2 + M^2} - \frac{3}{2}\frac{P^+}{\mathcal{P}^+}\Lambda^2 + \frac{1}{2}\frac{M^2\frac{P^+}{\mathcal{P}^+}\Lambda^2}{x_a\frac{P^+}{\mathcal{P}^+}\Lambda^2 + M^2} + 3\frac{M^2}{x_a}\ln\frac{M^2}{x_a\frac{P^+}{\mathcal{P}^+}\Lambda^2 + M^2}\right\}, \quad (5)$$

where $x_a = p_a^+/P^+$ is the longitudinal fraction of the momentum carried by the constituent under consideration, M is its mass, P^+ is the total longitudinal momentum of the state, $\mathcal{P}^+$ is the longitudinal scale required in the cutoff by dimensional arguments, and ϵ is an infrared cutoff which is to be taken to zero. The divergence in the effective one-body operator exactly cancels against a divergence in the effective two-body operator if the state is a color singlet [2].

The effective two-body operators have the following matrix elements between states containing a quark $\vec{p}_i$ and an antiquark:

$$-g_\Lambda^2\overline{u}(p_2,\sigma_2)\gamma^\mu u(p_1,\sigma_1)\overline{v}(k_2,\lambda_2)\gamma^\nu v(k_1,\lambda_1)\langle T_a T_b\rangle \times\left[\frac{1}{q^+}D_{\mu\nu}(q)\left(\frac{\theta(|D_1| - \Lambda^2/\mathcal{P}^+)\theta(|D_1| - |D_2|)}{D_1} + \frac{\theta(|D_2| - \Lambda^2/\mathcal{P}^+)\theta(|D_2| - |D_1|)}{D_2}\right)\right], \quad (6)$$

where $D_{\mu\nu}(q) = (q^{\perp 2}/q^{+2})\eta_\mu\eta_\nu + (1/q^+)(\eta_\mu q_\nu^\perp + \eta_\nu q_\mu^\perp) - g_{\mu\nu}^\perp$ is the gluon propagator in light-front gauge, D_1, D_2 are energy denominators $D_1 = p_1^- - p_2^- - q^-$ and $D_2 = k_2^- - k_1^- - q^-$. It has been shown that H_{eff} contains a logarithmic confining interaction in addition to the Coulomb interaction [2].

This is the output of the second order similarity transformation for $q\overline{q}$ matrix elements.

For the purpose of bound-state calculations, we split the effective Hamiltonian (3) into H_0, which is solved nonperturbatively, and $V = H_{\text{eff}} - H_0$. First, we make a nonrelativistic reduction of the effective Hamiltonian (3). In the nonrelativistic limit, the light-front scale $\Lambda^2/\mathcal{P}^+$ is naturally replaced by $\mathcal{L} \equiv (\Lambda^2/\mathcal{P}^+)(P^+/2M)$, where M is the mass of the heavy quark, and $\mathcal{L}$ carries the dimension of mass [3]. Further, light-front momenta are naturally replaced by center-of-mass equal-time momenta in the nonrelativistic limit [3].

The spin-independent part of the two-body effective interactions includes a short-range Coulomb potential and a rotationally noninvariant long-range logarithmic potential. The confining potential arises due to an incomplete cancellation of instantaneous gluon exchange, and it confines both quarks and gluons. The confining potential is a complicated function but in the nonrelativistic limit it can be double Fourier transformed (for longitudinal and transverse separation), and expanded in even Legendre polynomials [3].

For H_0 we choose the nonrelativistic reduction of the kinetic energy, the effective one-body operators, Coulomb potential and rotationally symmetric part of the confining potential with constituent masses. The Hamiltonian H_0 is

$$H_0 = 4M\left[-\frac{1}{2m}\vec{\nabla}^2 + \tilde{\Sigma} - \frac{C_F\alpha}{r} + \frac{C_F\alpha\mathcal{L}}{\pi}V_0(\mathcal{L}r)\right], \quad (7)$$

where m is the reduced mass, $\vec{r}$ is an equal-time separation between the quark and the antiquark, and $V_0(\mathcal{L}r)$ is the angular average of the confining potential generated by the similarity transformation. It depends only on the separation of the quarks:

$$V_0(\mathcal{L}r) = 2\ln\mathcal{R} - 2\text{Ci}(\mathcal{R}) + 4\frac{\text{Si}(\mathcal{R})}{\mathcal{R}} - 2\frac{(1-\cos\mathcal{R})}{\mathcal{R}^2} + 2\frac{\sin\mathcal{R}}{\mathcal{R}} - 5 + 2\gamma, \quad (8)$$

where $\mathcal{R} \equiv \mathcal{L}r$ and γ is Euler constant. $\tilde{\Sigma}$ contains the finite shift produced by the self-energies after subtracting terms needed to make the confining potential vanish at the origin:

$$\tilde{\Sigma} = \frac{\alpha C_F\mathcal{L}}{\pi}\left[\left(1 + \frac{3M}{4\mathcal{L}}\right)\ln\left(\frac{M}{\mathcal{L} + M}\right) + \frac{1}{4}\frac{M}{\mathcal{L} + M} + \frac{5}{4}\right]. \quad (9)$$

The remaining part of the effective Hamiltonian, V, contains, among other terms, emission and absorption of low energy gluons. Interactions that change particle number enter at second-order bound-state perturbation theory, which requires solutions to the nonperturbative three-body bound-state problem.

V also contains a rotationally noninvariant part of the effective confining potential. Our choice of H_0 does not lead to any first-order corrections to S states due to the

rotationally noninvariant part of the potential, and for any $l \neq 0$ state it minimizes the number of terms which give nonzero corrections, thus making the calculations easier.

The spin-dependent part of the two-body effective operators is included in V, and it is treated in first-order bound-state perturbation theory. We will consider only the spin-spin hyperfine splitting in the ground state, because it can be calculated using the lowest order effective Hamiltonian [6]. After a change of the spinor basis [6], the spin-spin part of the two-body effective interactions is

$$v_{\text{spin}} = 4(2M)^2 \alpha C_F \mathcal{L}^3 \left[\frac{8\pi}{3} \delta^3(\vec{\mathcal{R}}) + \frac{2}{\pi} f(\vec{\mathcal{R}}) \right] \times \frac{1}{4M^2} \vec{\sigma}_a \cdot \vec{\sigma}_b \,. \quad (10)$$

The function $f(\vec{\mathcal{R}})$ is rotationally noninvariant with respect to the angle θ. Its angular average is

$$\langle f(\mathcal{R}) \rangle = \frac{2}{3} \left[\frac{\sin \mathcal{R}}{\mathcal{R}^3} - \frac{2(1 - \cos \mathcal{R})}{\mathcal{R}^4} \right]. \quad (11)$$

This completes our discussion on H_0 and V.

We now want to solve nonperturbatively the eigenvalue problem for H_0:

$$H_0 |P\rangle_\Lambda = \mathcal{M}^2 |P\rangle_\Lambda \,, \quad (12)$$

where $\mathcal{M}^2$ is the invariant mass of the bound state. We assume that the scale Λ is small enough so that the state is dominated by its $q\overline{q}$ component [3], i.e.,

$$|P\rangle_\Lambda = \int \frac{d^2\kappa^\perp dx}{2(2\pi)^3 \sqrt{x(1-x)}} \psi(\kappa^\perp, x) b^\dagger d^\dagger |0\rangle \,. \quad (13)$$

Let the mass of the bound state be

$$\mathcal{M}^2 = (2M)^2 + 4ME \,, \quad (14)$$

which defines E.

The eigenvalue problem for the Hamiltonian H_0 leads to a dimensionless Schrödinger equation [3]:

$$\left[-\frac{d^2}{d\vec{\mathcal{R}}^2} + c\left(\frac{1}{\pi} V_{\text{conf}}(\vec{\mathcal{R}}) + V_{\text{Coul}}(\mathcal{R}) \right) \right] \psi(\vec{\mathcal{R}}) = e\psi(\vec{\mathcal{R}}) \,, \quad (15)$$

where

$$c \equiv \frac{2m\alpha C_F}{\mathcal{L}} \,, \quad (16)$$

$$e \equiv \frac{2m(E - \tilde{\Sigma})}{\mathcal{L}^2} \,. \quad (17)$$

In Ref. [3] we show the dimensionless eigenvalue e for a few low-lying states, for c ranging up to 1. From the ratio of the splittings between $1S$ and $1P$, and $1P$ and $2S$ charmonium states, we find that c should be around 0.6. For charmonium, we find values of the quark mass M_c, the cutoff Λ, and $\alpha = g^2/4\pi$ so that (i) $c = 0.6$ (or, equivalently, the ratio of the $1S$-$1P$ splitting to the $1P$-$2S$ splitting is roughly correct), (ii) the mass of the ground state is $\mathcal{M}_{1S} = 3.0$ GeV, and (iii) the mass of the lowest lying P state is $\mathcal{M}_{1P} = 3.5$ GeV. Note that these values are reasonable approximations given the magnitude of known corrections. We obtain $M_c = 1.5$ GeV, $\alpha = 0.5$, and $\Lambda = 1.7$ GeV. Similarly, for bottomonium we require $c = 1.0$, $\mathcal{M}_{1S} = 9.4$ GeV, and $\mathcal{M}_{1P} = 9.9$ GeV leading to a bottom quark mass $M_b = 4.8$ GeV, $\alpha = 0.4$, and $\Lambda = 3.5$ GeV. It is important to note that this coupling need not run exactly like $\alpha_{\overline{\text{MS}}}$ or α_{lattice}.

With the parameters fixed, we can predict the hyperfine splitting in the charmonium ground state using Eq. (10). The function $f(\vec{\mathcal{R}})$ is rotationally noninvariant, but at $c = 0.6$ the violation of rotational symmetry is negligible. We predict that the splitting between the ground state vector and singlet in charmonium is 0.13 GeV, in reasonable agreement with experiment (0.118 ± 0.002 GeV).

Next, we evaluate corrections due to the rotationally noninvariant part of the confining potential. There are first-order corrections to the P state, and the S state is corrected in second-order bound-state perturbation theory. Corrections to the ground state are consistently a few percent even for c as small as 0.1. Corrections to the excited states at $c = 0.6$ and $c = 1.0$ are about 30%. This is a reasonable starting point because corrections of one higher power of α are of this same order.

In conclusion, the logarithmic confining potential which arises at second order is a promising starting point for QCD calculations. Corrections to the energy levels due to rotational symmetry violating terms in this potential are negligible for the ground state, and for the lowest excited states they are small enough that corrections from higher-order terms may restore rotational symmetry. This calculation is not intended to compete with phenomenological constituent quark model calculations. It is intended as an initial crude step toward an accurate first principles, light-front QCD calculation.

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