Gap Independence and Lacunarity in Percolation Clusters

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The gaps between occupied sites on linear cuts of two and three dimensional critical percolation clusters are found to be closely described as statistically *independent*, with a universal scaling distribution close to that of positive Lévy flights. The moments of the mass distribution of Lévy flights obey $\langle m^k\rangle/\langle m\rangle^k = k![\Gamma(\alpha+1)]^k/\Gamma(k\alpha+1)$, where α is their fractal dimension. Our data on linear cuts of critical percolation clusters are consistent (within the numerical error bars) with these predictions. The property of statistical independence of the gaps characterizes the *lacunarity* of the percolation clusters as being *neutral*. [S0031-9007(96)00801-0]

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Nature contains many random fractal structures [1], and much effort has been devoted to identifying good ways to characterize them and to divide them into universality classes. The fractal dimension D, which describes the scaling of the mass M within a volume of size L, via $M \propto L^D$, provides one such characteristic. However, many different structures share the same value of D but look very different from each other ([1], Chaps. 34 and 35 [2,3]). A more detailed characterization involves cutting a fractal structure which is embedded in a d dimensional Euclidean space with a one dimensional line. Self-similarity implies that the mass of the fractal dust of points on the linear cut scales as $m = AL^\alpha$, where $\alpha = D - d + 1 < 1$ is the fractal dimension of the cut [1]. Although the exponent α characterizes the scale in all the measurements, the amplitude A depends on the process (A is not the same for all fractals for which the cut has dimension α), fluctuates among different random realizations and between different linear cuts, and may even oscillate as a function of $\ln L$ for a given realization [4]. These variations result from the presence of empty holes of different sizes, and are associated with the concept of *lacunarity* [1].

A nontrivial consequence of self-similarity is that the moments of m scale as

$$\langle m^k\rangle = \mu_k L^{k\alpha}, \quad (1)$$

with the *unifractal* exponents $k\alpha$ and with $\mu_k = \langle A^k\rangle$. The cumulants $\langle A^k\rangle_c$ and, particularly, $\langle \Delta m^2\rangle/\langle m\rangle^2 \equiv \langle A^2\rangle_c/\langle A\rangle^2 = \mu_2/\mu_1^2 - 1$ have been used to quantify mass lacunarity: large (small) values of this ratio correspond to a less (more) uniform mass distribution [1,4]. Fractal models with small mass lacunarity have been shown to relate to standard analytic continuations of Euclidean dimensions [5].

Starting from a point on the line which belongs to the structure, consider the length of the nearest gap, that is, the distance t_1 to the next such point along the line, and similarly the lengths t_i between consecutive more distant gaps. By a general theorem on fractals, the distribution of t_i is $\Pr\{t_i > t\} \sim t^{-\alpha}$ ([1], p. 78). Our simulations on critical percolation clusters, presented below, confirm this prediction. *Lacunarity* involves the *complete* distribution of points along the linear cut, including the possible *interdependence* between successive gaps. Statistically independent gaps characterize the Lévy dust (stopovers of a positive Lévy flight that always moves to the right). This case was selected [2,3,6] as defining a *neutral lacunarity*, i.e., the boundary between *high and low lacunarity*, as reflected, e.g., by negative and positive antipodal correlations, respectively [3]. Since the gap lengths of fractal dusts have infinite expectation, correlation is not usable in their study and special methods were brought to bear. For critical percolation clusters, a test of antipodal correlations [3] favored asymptotic independence, which was approached slowly (from the negative side) as the system size increased. The goal of this Letter is to report two additional tests that support the notion that a cut through a critical percolation cluster is modeled well by a Lévy dust of dimension $\alpha = D - d + 1$, where D is the fractal dimension of the cluster. Just like Lévy dusts, critical clusters exhibit *long range correlations.* Therefore, the investigations in [3] and in this paper demonstrate an important new feature of the correlations present in percolation clusters: within small deviations—which may be due to sampling fluctuations (e.g., averages being dominated by rare events), slow finite size convergence or other systematic errors—*they are compatible with independent gaps.* This conclusion yields a variety of quantitative predictions involving the distributions of these gaps.

To study dependence, we renormalize the gaps by putting q successive gaps together. The length of a "q gap" is $x_q = \sum_{i=1}^q t_i \geq q$, and we consider the probability $N_q(s)$ that $x_q = s$. If the $\{t_i\}$'s were *independent* identically distributed random variables, then a generalization of the central limit theorem implies that, for $q, s \gg 1$, $N_q(s)$ approaches a *stable distribution* [7,8] described by the scaling form

$$N_q(s) = (\gamma q)^{-1/\alpha} F_\alpha[(s - \delta q)(\gamma q)^{-1/\alpha}], \quad \text{as } q, s \gg 1. \quad (2)$$

Here, the scale factors γ and δ are nonuniversal, but F_α is a universal scaling function determined completely by the fractal dimension α. When the independent variables t_i have a finite variance, then the central limit theorem yields the Gaussian distribution, i.e., Eq. (2) with $\alpha = 2$. In contrast, for self-similar (fractal) dusts with $\alpha < 1$, $N_1(s)$ decays with the power law $s^{-\alpha-1}$. In this case we expect N_q also to decay as $s^{-\alpha-1}$ for sufficiently large s, and to have some nontrivial structure when s becomes of order $q^{1/\alpha}$ [7,8]. Specifically, in this case one has

$$F_\alpha(u) = \frac{1}{\pi u} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sin(\pi k \alpha) \Gamma(k\alpha + 1) u^{-k\alpha}, \quad (3)$$

where $\Gamma(x)$ is the gamma function. These results also imply that, for a stable distribution, the mass amplitudes A obey the Mittag-Leffler distribution [6,9] with the universal amplitude ratios

$$\Delta_k \equiv \mu_k / \mu_1^k = k! [\Gamma(\alpha + 1)]^k / \Gamma(k\alpha + 1). \quad (4)$$

The present Letter considers critical percolation clusters, at the percolation threshold p_c, which have become an excellent test ground for studying physics on random fractal structures [10]. Two of the present authors have measured the antipodal correlations in two dimensional (2D) percolation clusters, and found them to be slightly negative and decaying to zero for large samples [3]. This decay was interpreted as indicating asymptotic neutral lacunarity. However, this measurement is somewhat indirect; a much clearer check of the dependence along the linear cut is obtained by comparing the mass distribution with the predictions based on stable distributions for the reference positive Lévy flight. The present Letter reports on measurements of $N_q(s)$ and of the Δ_k's for linear cuts of critical percolation clusters in two and three dimensions; they turn out to be closely described by Eqs. (1)–(4). This is strong evidence that consecutive steps along linear cuts of such clusters are indistinguishable from *independent.* Thus it seems that the universality class of stable distributions, which includes the Lévy flights, can also describe critical percolation clusters. This opens the possibility of using this universal distribution to calculate other percolation properties.

Our numerical work began by testing Eqs. (1) and (4). We generated the percolation clusters at p_c using the Leath algorithm [11], starting with a single occupied site in the center of the lattice of size $(2L_{\max} + 1)^d$, and continuing the growth until the cluster touched one of the boundaries. In order to generate large statistics, several lines, both horizontal and vertical, were analyzed for each realization of the cluster, collecting data only for those horizontal (vertical) lines for which the point at $x = 0$ ($y = 0$) (zeroth incident) was occupied. For each such line we counted the mass m as the number of the occupied sites within a linear segment of length L, but excluding the zeroth incident point.

Figure 1 shows the average mass for cuts on the square, triangular, and 3D simple cubic lattices, at criticality (that is, for $p_c = 0.592\,746$ [12], 0.5, and 0.3116 [10], respectively), as a function of the linear size L (lattices with $30\,001^2$ and 1025^3 sites). Except for finite size effects when L approaches $L_{\max}$, the data are consistent with Eq. (1), and the slopes agree with $\alpha = D - d + 1$ for percolation clusters, for which one expects the asymptotic values $D = 91/48$, 2.53 in 2D, 3D [10], as shown by the dashed lines. Quantitative measurements of the logarithmic local slopes allow errors of ± 0.006 and ± 0.02 on the measured α's. Since we believe in the asymptotic value $\alpha = D - d + 1$, we attribute this scatter to numerical fluctuations, e.g., due to finite sampling. Similar behavior, with exponent $k\alpha$, was observed for the higher moments of the mass, confirming the unifractal mass distribution of the cuts (data not shown).

The analysis of the moment ratios Δ_k turned out to be complicated because of the finite size effects: we need to extrapolate both L and $L_{\max}$ to infinity, keeping the ratio $L/L_{\max}$ very small. In practice, both $1/L$ and $L/L_{\max}$ are finite, and the competition between them results in "U"-shaped curves when the data for Δ_k are plotted as a function of $1/L$ (as done in the insets in Fig. 2). We analyzed these data assuming $\Delta_k(L, L_{\max}) = \Delta_k + a_k/L^x + b_k(L/L_{\max})^y$; this simple scaling predicts that the minima scale as $\min[\Delta_k(L, L_{\max})] = \Delta_k + a'_k L_{\max}^{-z}$, where $z = xy/(x + y)$. We find our data to be compatible with the theoretically reasonable values $x = y = 1$ and therefore also with $z = 1/2$. For example, the extrapolation of Δ_2 using $z = 1/2$ is shown in Fig. 2(a) [2(b)] for the square [triangular] lattice. We also tried several alternative analyses, and all gave similar results, leading to our final estimates of $\Delta_k^{\text{2D perc}} = 1.096 \pm 0.015$, $1.26 \pm 0.04, 1.50 \pm 0.06$ for $k = 2, 3, 4$, respectively. The error bars account for the difference between the two lattices and especially the spread of the extrapolation results between the different methods of analysis. The estimate ranges include the theoretical predictions for the stable distribution for this value of α [see Eq. (4)], for which $\Delta_k = 1.108, 1.292, 1.561$, especially if one notes that the error bars on α also imply error bars on these "theoretical" values. (For example, the range of "measured" α implies values of Δ_2 between 1.101 and 1.114.) Similar analysis of the 3D data gives $\Delta_k^{\text{3D perc}} = 1.53 \pm 0.04, 2.90 \pm 0.20, 6.3 \pm 0.5$, for $k = 2, 3, 4$, again including the moments of the stable distributions: for $\alpha = 0.53$, Eq. (4) yields $\Delta_k = 1.534, 2.957, 6.648$. The errors in α again add some uncertainty to the latter values. Given the difficulties in finite size extrapolation, we conclude that our data show no observable deviations from those of Lévy flights.

We next turn to the distribution $N_q(s)$. We first note that $F_\alpha(u) \propto u^{-\alpha-1}$ for $u \gg 1$. (This can be seen heuristically, e.g., by assuming a power law decay for F_α, calculating $\langle q \rangle = \langle m \rangle = \sum_{q=1}^{L} q N_q(L)/[\sum_{q=1}^{L} N_q(L)]$ with fixed, large L, and requiring that $\langle m \rangle \propto L^\alpha$.) For very small u, the probability to find q consecutive occupied sites on the line equals approximately p_c^q, with a combinatorial prefactor which may involve a power of q or at least a multiplicative constant. Keeping only the exponential accuracy, one can write $N_q(q) \approx p_c^q$. Using the scaling form of Eq. (2), a change in variables yields the asymptotics for $u \ll 1$,

$$\ln F_\alpha(u) \approx -(Cu)^{\alpha/(\alpha-1)}, \tag{5}$$

with $C = [-\ln(p_c)/\gamma]^{(\alpha-1)/\alpha}\gamma/(1-\delta)$. It is satisfactory to note that these percolation arguments capture essentially the correct asymptotics for the general family of the stable distributions F_α [7,13].

To check Eqs. (2) and (3) quantitatively, we calculated first the distribution $N_q(s)$ for percolation cluster cuts on the square and triangular lattices. Figures 3(a) and 3(b) show the excellent data collapse of $N_q(s)(\gamma q)^{1/\alpha}$ for these systems, as predicted in Eq. (2). Furthermore, the observed data collapse also supports both asymptotics as discussed above [in 3D, in Fig. 3(c), the small u asymptotic form contains an additive constant to Eq. (5), representing the leading correction to that limit]. In this figure we have chosen scale and location parameters (δ and γ) so that the measured scaling function matches the stable distribution F_α. Indeed, choosing $\gamma = 1.8$ and $\delta = -0.45$ ($\gamma = 3.1$, $\delta = -1.7$) for the square (triangular) lattice gives a reasonable fit to F_α [see Eq. (3) and solid lines in Fig. 3]. In particular, the data for the triangular lattice fit excellently with the stable distribution. These values of γ and δ were fixed using the eye, by choosing the best looking fit with F_α. Some quantitative measure of the quality of the fits, and especially of the consistency of the theory, can be gained by noting that the universality of F_α implies the universality of the

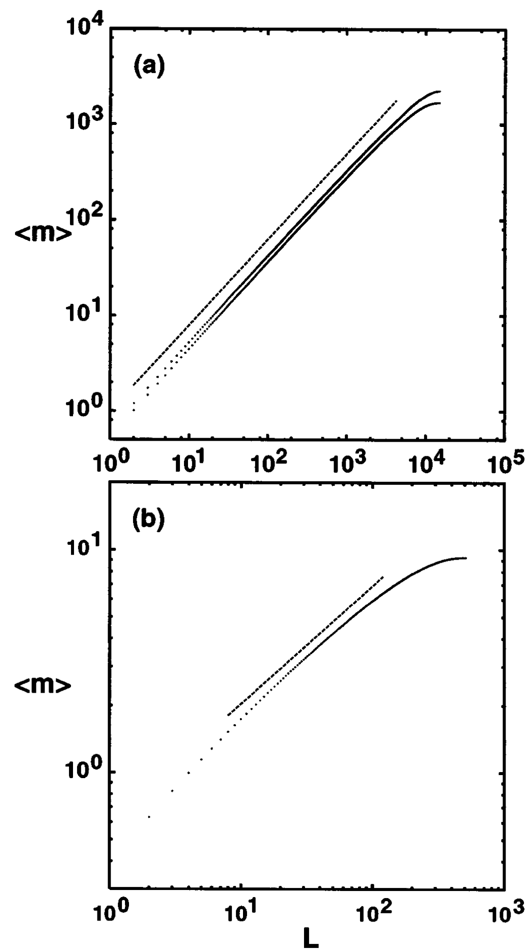

FIG. 1. Log-log plot of the average mass (dots) versus linear size L for the (a) square (upper points) and triangular (lower points) lattices, (b) 3D simple cubic lattice. The dashed lines in (a) and (b) have slopes 43/48 and 0.53, respectively.

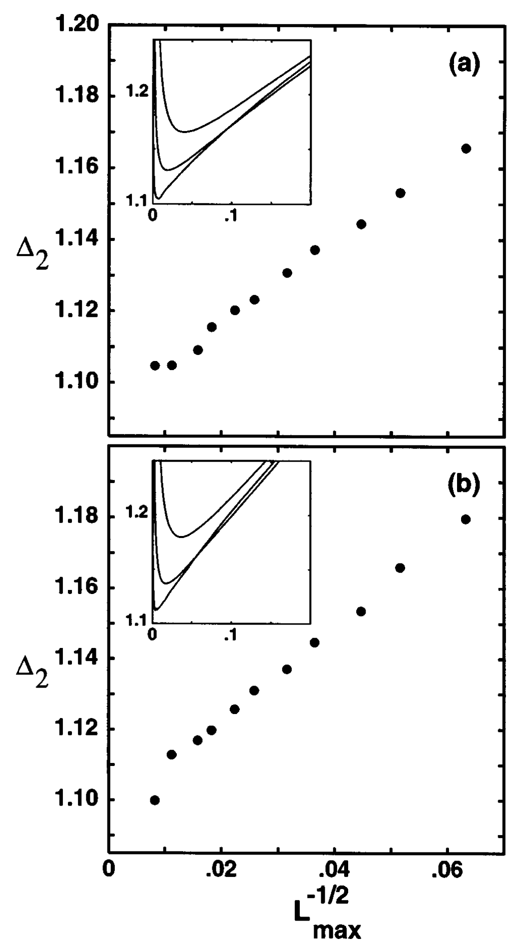

FIG. 2. The extrapolation of Δ_2 for the (a) square, (b) triangular lattice, by plotting the minima of Δ_2 versus $L_{\max}^{-1/2}$. The insets show the "U"-shaped raw Δ_2 plotted versus $1/L$ for $L_{\max} = 250, 1000, 8000$ (from top to bottom).

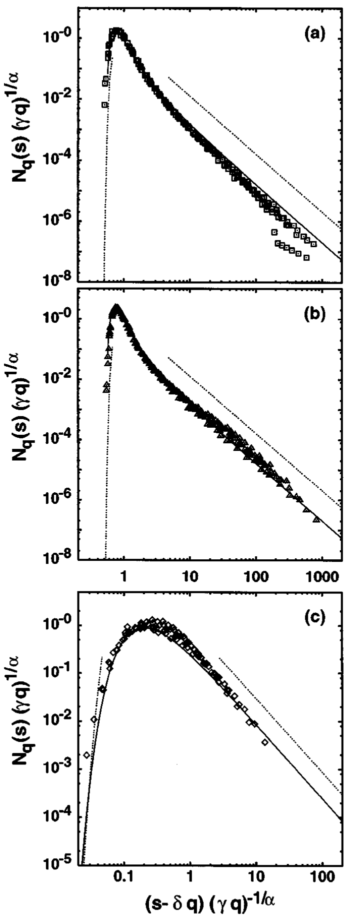

FIG. 3. The scaled distribution $(\gamma q)^{1/\alpha} N_q(s)$, $\alpha = D - d + 1$, measured from the percolation cluster at p_c on the (a) square, (b) triangular, and (c) 3D simple cubic lattice. Markers denote different values of q: $q = 3, 5, \ldots, 19$ in (a), (b), and $q = 3, 4, \ldots, 11$ in (c). Note also different scales in (a), (b), and (c). The dashed lines show the asymptotic behavior for the scaling function: the right-hand side line has slope $-\alpha - 1$, and the line on the left represents Eq. (5). The solid line is the stable distribution of Eq. (3).

coefficient C. Indeed, the values of C for the square ($C = 1.43$) and triangular ($C = 1.37$) lattices are within 4% of each other. Furthermore, these values are also close to the exact value for Lévy flights, $C = 1.452$.

Our simple cubic lattice results yield similar data collapse for $N_q(s)$, as depicted in Fig. 3(c). The quality of the data collapse is less good than in 2D, because we had to use relatively small system sizes ($L_{\max} \leq 512$). This may also be the reason for the worse agreement with the stable distribution asymptotics. Despite these apparent finite size effects the observed data collapse for 3D percolation, as well as the fit with the stable distribution, seems consistent with Eqs. (2) and (3).

In conclusion, we find that linear cuts of two and three dimensional critical percolation clusters have almost independent gaps, hence may be described using the same universal class as positive Lévy dusts, with the corresponding stable distribution, Eqs. (1)–(4). This suggests that the lacunarity of percolation cluster is very closely neutral. It would be interesting to check the conjecture that this neutrality is, in fact, exact. We hope that similar methods can be used to calculate other percolation properties, and that this Letter will stimulate similar studies on other fractal structures, and more numerical and theoretical work directed to understanding the origin of this independence.

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