Universal Scaling Functions for Numbers of Percolating Clusters on Planar Lattices

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Using a histogram Monte Carlo method and nonuniversal metric factors of a recent Letter [Phys. Rev. Lett. **75**, 193 (1995)], we find that the probability for the appearance of n, $n = 1, 2, \ldots$, top to bottom percolating clusters on finite square, planar triangular, and honeycomb lattices falls on the same universal scaling functions, which show interesting behavior as the aspect ratio of the lattice increases. Our results suggest many interesting problems for further research. [S0031-9007(96)00574-1]

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Percolation is an active research subject in recent years and is related to many interesting problems [1]. Percolation models have been used to clarify or formulate many important ideas and useful algorithms, e.g., corrections to scaling [2,3], universality of critical existence probability [2–4], histogram Monte Carlo methods [5,6], cell-to-cell renormalization group method [7–10], universal scaling functions and nonuniversal metric factors [11,12], etc. In this Letter, we use a histogram Monte Carlo simulation method (HMCSM) [5] to study an interesting and not well studied quantity: the probability W_n for the appearance of $n, n = 1, 2, \ldots$, top to bottom percolating clusters on finite lattices. We find that W_n has very good finite-size scaling behavior near the critical probability p_c [13]. Using nonuniversal metric factors [14] of a recent Letter [11], we find that W_n for bond and site percolations on finite square (sq), planar triangular (pt), and honeycomb (hc) lattices fall on the same universal scaling functions. The scaling functions show interesting behavior as the aspect ratio of the lattice increases. Our results suggest many interesting theoretical and experimental problems for further research, including conformal theory [15], critical phenomena of lattice models [16,17] and hard disks or spheres [18], transport of fluid and current through random mediums, quantum Hall effects [19], etc.

Important quantities in traditional studies of a percolation problem on a lattice G of linear dimensions L include the existence probability $E_p(G, p)$ and the percolation probability $P(G, p)$ with p being the bond or site occupation probability. Here $E_p(G, p)$ is the probability [5–9] that the system percolates and $P(G, p)$ is the probability that a given lattice site belongs to a percolating cluster. In the free boundary conditions and in the limit $L \to \infty$, it has been found that for site and bond percolation on the $L \times L$ sq lattice, $E_p(G, p_c) = 0.5$ [2–4] and it has been proposed by Langlands, Pichet, Pouliot, and Saint-Aubin (LPPS) [4] that for bond and site percolation on the pt and hc lattices with aspect ratios $\sqrt{3}/2$ and $\sqrt{3}$, respectively, $E_p(G, p_c)$ is also equal to 0.5. Using a HMCSM [5] and the aspect ratios of LPPS [4], Hu, Lin, and Chen (HLC) have found that all scaled data of E_p and P as a function of scaling variable x fall on the same universal scaling functions $F(x)$ and $S(x)$, respectively, where $x = D_1(p - p_c)L^{1/\nu}$ with D_1 being a model dependent nonuniversal metric factor and ν being a correlation length exponent. HLC also found that D_1 is independent of the boundary conditions [11] and the extra factor multiplied on aspect ratios of all lattices [12]. In this Letter, we first use the HMCSM [5,13] to evaluate the probability $W_n(L_1, L_2, p)$ for the appearance of n top-to-bottom percolating clusters of bond percolation on finite $L_1 \times L_2$ sq [20] lattices with a linear dimension L_1 in the horizontal direction and a linear dimension L_2 in the vertical direction. When we plot W_n as a function of the scaling variable $z = (p - p_c)L^{1/\nu}$, all data of the same aspect ratio L_1/L_2 fall on the same scaling function. Using the nonuniversal metric factor D_1 of [11], we then show that W_n for bond and site percolations on sq, pt, and hc lattices with aspect ratios L_1/L_2, $\sqrt{3}L_1/2L_2$, and $\sqrt{3}L_1/L_2$ have universal scaling functions when they are plotted as functions of x, where $x = D_1 z$.

Here we briefly review the HMCSM for calculating $W_n(L_1, L_2, p)$ of the bond percolation on sq lattices [5,13]. The extension to other lattices and site percolation is straightforward. In the bond percolation on a $L_1 \times L_2$ sq [20] lattice G of N sites and E bonds, $N = L_1 \times L_2$, each bond of G is occupied with a probability p, where $0 \le p \le 1$. The lattice G has free and periodic boundary conditions in the vertical and horizontal directions, respectively. A cluster which extends from the top row of G to the bottom row is a percolating cluster. The subgraph which contains at least one percolating cluster is a percolating subgraph and denoted by G'_p; otherwise it is a nonpercolating subgraph. The percolating subgraph with n percolating clusters is denoted by G'_n. Now we have the definition

$$W_n(L_1, L_2, p) = \sum_{G'_n \subseteq G} p^{b(G'_n)}(1 - p)^{E - b(G'_n)}, \quad (1)$$

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where $b(G'_n)$ is the number of occupied bonds in G'_n. The summation in (1) is over all subgraphs G'_n of G. In the HMCSM, we choose w different values of p. For a given $p = p_j$, $1 \leq j \leq w$, we generate N_R different subgraphs G'. The data obtained from the wN_R different G' are then used to construct arrays of numbers with elements $N_p(b)$, $N_f(b)$, and $N_n(b)$, $0 \leq b \leq E$, which are, respectively, the total numbers of percolating subgraphs with b occupied bonds, nonpercolating subgraphs with b occupied bonds, and the number of percolating subgraphs with b occupied bonds and n percolating clusters. In the large number of simulations, the probability W_n at any value of the bond occupation probability p can then be calculated approximately from the following equation [5,13]:

$$W_n(L_1, L_2, p) = \sum_{b=0}^{E} p^b (1 - p)^{E-b} C_b^E \frac{N_n(b)}{N_p(b) + N_f(b)}, \tag{2}$$

where $C_b^E = E!/(E - b)!\,b!$. It is obvious that $E_p = \sum_{n=1}^{\infty} W_n$.

We first use (2) to evaluate W_n for bond percolation on 128×32, 256×64, and 512×128 sq [20] lattices. The results are shown in Fig. 1(a), where $W_0 = 1 - E_p$. Using the exact values of ν and p_c [1], we obtain W_n as a function of $z = (p - p_c)L^{1/\nu}$. The results are shown in Fig. 1(b), where very good scaling behavior of W_n is observed and the corresponding scaling function is denoted by $F_n(R, z)$ with $R = L_1/L_2$. Figure 1(b) shows that $F_n(R, z)$ for $n \geq 2$ is a symmetric function of z.

In [11], HLC studied bond and site percolations on a 512×512 sq lattice, a 433×500 pt lattice, and a 433×250 hc lattice to obtain universal scaling functions and nonuniversal metric factors, e.g., D_1, which means that HLC used 433/500 and 433/250 to approximate, respectively, $\sqrt{3}/2$ and $\sqrt{3}$ considered by LPPS [4]. Now we calculate $W_n(L_1, L_2, p)$ for bond and site percolations on a 512×128 sq lattice, an 866×250 pt lattice, and an 866×125 hc lattice, which means that aspect ratios of all lattices of [11] are multiplied by 4. When the calculated W_n for all lattices, shown in Fig. 2(a), are plotted as a function of $x = D_1(p - p_c)L^{1/\nu}$ with D_1 taken from [11], all calculated results for each n fall nicely on the same universal scaling functions, $U_n(x)$, shown in Fig. 2(b). This is additional evidence that D_1 is independent of the extra factor for aspect ratios, which is 4 in this case. The universality of scaling functions for $W_n(L_1, L_2, p)$ means that the results obtained from sq lattices, e.g., those to be shown below, may be applied to corresponding pt and hc lattices.

We have also calculated $F_n(R, z)$ for sq lattices for various values of R, which are shown in Figs. 3(a) and 3(b) for $n = 1$ and 2, respectively. $F_n(R, 0)$ as a function of R for $n = 0, 1, \ldots, 6$ is shown in Fig. 4(a). Figure 4(a) shows that when R increases, $F_n(R, 0)$ for $n > 0$ first increases to a certain maximum, then begins to decrease. Figure 3 shows that when F_n begins to decrease it also

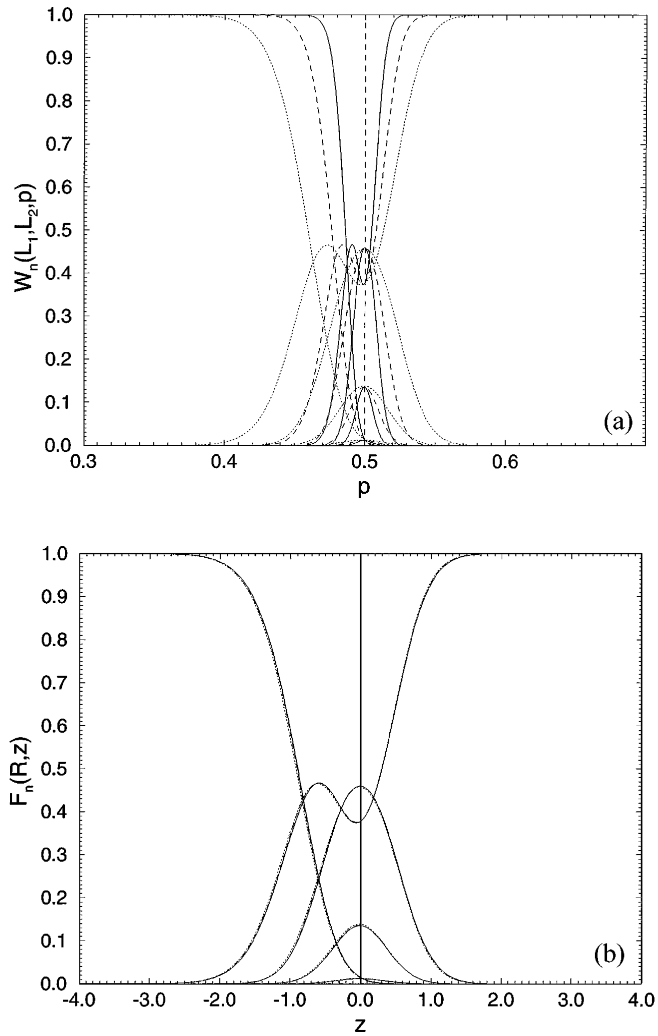

FIG. 1. (a) $W_n(L_1, L_2, p)$ for bond percolation on 128×32, 256×64, and 512×128 lattices, which are represented by dotted, dashed, and solid lines, respectively. (b) The data of (a) are plotted as a function of $z = (p - p_c)L^{1/\nu}$. The scaling function for $W_n(L_1, L_2, p)$ is denoted by $F_n(R, z)$, where $R = L_1/L_2$. The monotonic decreasing function is for $F_0(R, z)$. The S shaped curve is for $F_1(R, z)$. The bell shaped curves from top to bottom are for $F_n(R, z)$ with n being 2, 3, and 4, respectively.

begins to develop a valley with a minimum at $z = 0$. When R continues to decrease, the valley becomes deeper and deeper. From $F_n(R, z)$, we may calculate the average number of percolating clusters $C(R, z)$ via $C(R, z) = \sum_{n=1}^{\infty} F_n(R, z)n$. $C(R, 0)$ as a function of R is shown by a solid line in Fig. 4(b). We have extended all of the above calculations to sq lattices with free boundary conditions in both horizontal and vertical directions. Now F_n for $n \geq 2$ is not a symmetric function of z, the maximum or the minimum of F_n moves to $z > 0$. $C(R, 0)$ as a function of R in this case is shown by a dotted line in Fig. 4(b). Figure 4(b) shows that, for large R, $C(R, 0)$ increases linearly with R for both periodic and free boundary conditions in the horizontal direction and two cases have the same slope. It is of interest to know that for a given

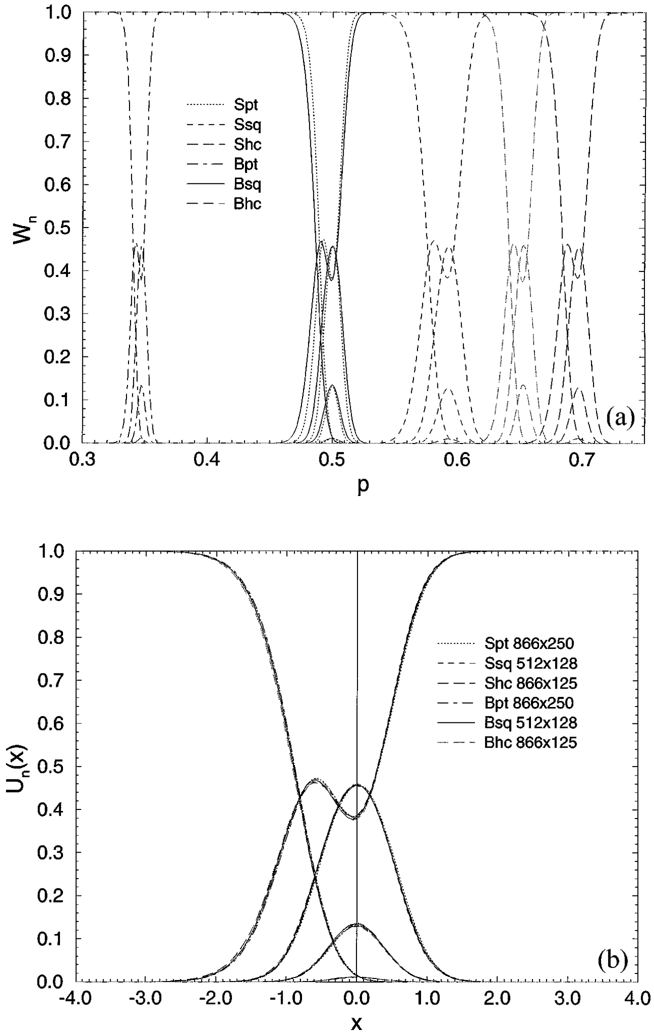

FIG. 2. (a) W_n for bond and site percolations on 866×250 pt, 512×128 sq, and 866×125 hc lattices. (b) The data of (a) are plotted as a function of $x = D_1(p - p_c)L^{1/\nu}$. The universal scaling function for W_n is denoted by $U_n(x)$.

large R, $C(R, 0)$ for the free boundary condition is larger than $C(R, 0)$ for the periodic boundary condition and such an effect persists even for very large R.

Many results presented above may be understood from the general theory of critical phenomena and finite-size scaling. The arguments used to obtain the universality of finite-size scaling functions for E_p [3] may be extended to arrive at the universality of finite-size scaling functions for W_n. At the critical point p_c, the correlation length goes to ∞; therefore a finite system at p_c will feel strongly the effects of boundary and different boundary conditions give quite different $C(R, 0)$, which is similar to the case discussed in [8]. For large R, each percolating cluster has a fixed average linear dimension in the horizontal direction; thus $C(R, 0)$ increases linearly with R.

We expect that the scaling behavior of W_n found in this Letter may be extended to higher dimensions and correlated and continuum percolation problems [16–18]. Our results suggest many interesting problems for further research, including (1) scaling behavior of $F_n(R, 0)$ for large values of n, (2) calculation of $F_n(R, 0)$ by conformal theory [15], (3) calculation of $F_n(R, z)$ for correlated percolation models corresponding to Ising-type spin models [16] and hard-core particle models [17,18], which is useful for understanding the magnetic susceptibility in finite systems [16], (4) connection between $F_n(R, z)$ and transport of fluid and current through random mediums, and (5) quantum Hall effects [19] to be discussed in the next paragraph.

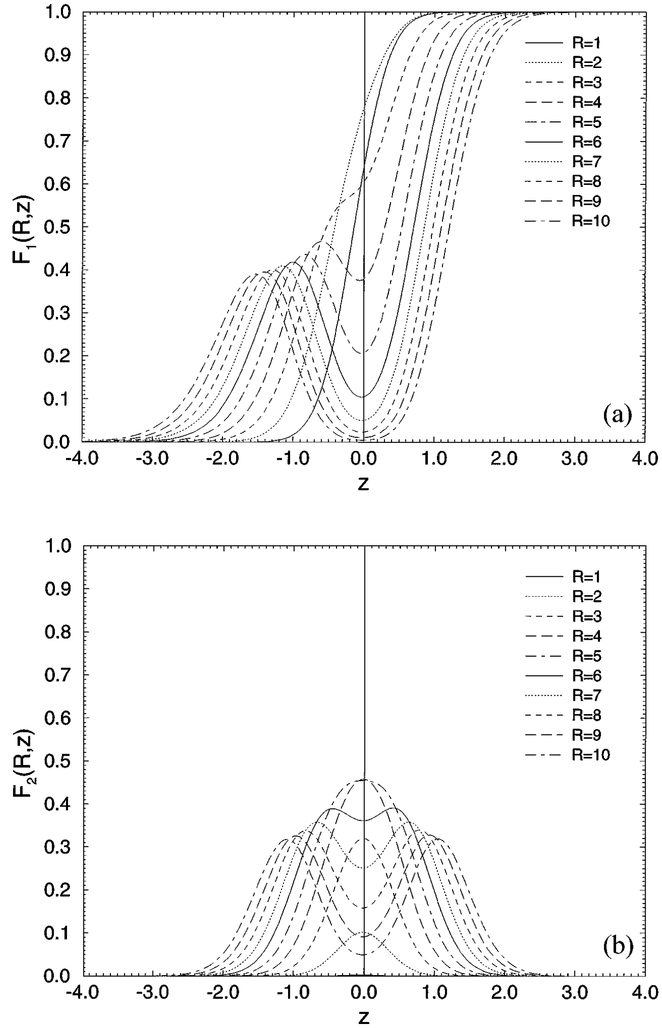

FIG. 3. $F_n(R, z)$ for $R = L_1/L_2 = 1, 2, 3, \ldots, 10$. (a) $n = 1$, (b) $n = 2$.

In a recent paper [19], Ruzin, Cooper, and Halperin (RCH) proposed that $\sigma_{xx}^{\max}$ in the quantum Hall effects is proportional to the number k of percolating clusters in the sample. The nice scaling behavior of $W_n(L_1, L_2, p)$ suggests that $F_n(R, z)$ obtained from small systems may be applied to large systems. This is a good starting point for the effort to compare simulation results with experimental data. To do such a comparison, one should pay attention to the following points: (a) By a conformal transformation, our $L_1 \times L_2$ system with free and periodic boundary conditions in vertical and horizontal directions, respectively, may be mapped into a Corbino disk, the geometry for the

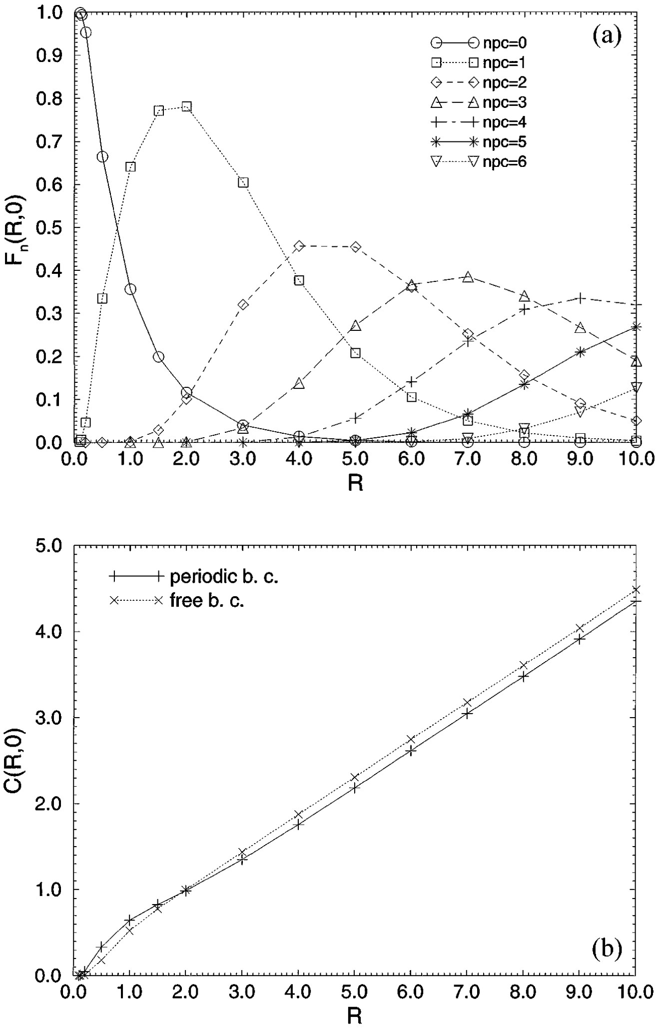

FIG. 4. (a) $F_n(R,0)$ as a function of $R = L_1/L_2$ for a number of percolating clusters (npc) runs from 0 to 6. Here the lattices have horizontal periodic boundary conditions. (b) $C(R,0)$ as a function of $R = L_1/L_2$ for the periodic (solid line) and free (dotted line) boundary conditions in the horizontal direction.

sample used in many experiments of the quantum Hall effect [19]. Therefore, at the critical point, where conformal invariance is valid, our results for rectangular domains may be applied to corresponding Corbino disks. (b) In a percolation problem, we may consider occupied bonds (or sites) to be conducting and nonoccupied bonds (or sites) to be nonconducting. In the general case, we may have n_p percolating conducting clusters and n_n percolating nonconducting clusters in a system. In this Letter we calculate only n_p. The number of percolating clusters, k, in the RCH theory [19] is given by $k = \min(n_p, n_n)$. For the Corbino disk and the corresponding lattice discussed in (a), $n_n = n_p$ for $n_p \geq 2$ [21] and the results of this Letter for $n \geq 2$ may be used to compare with experimental data. However, for $n_p = 1$ it is possible that $n_n = 0$ [22] and our result for $n_p = 1$ is an upper bound of the result for $k = 1$. (c) Our results are obtained from random percolation models. To compare our results with experimental data, one should make sure that the experimental system may be well represented by a random percolation model.

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[20] Here "square" means a primitive unit cell of the lattice is square rather than $L_1 = L_2$.
[21] When $n \geq 2$, two percolating conducting clusters are always separated by a percolating nonconducting cluster; therefore $n_p = n_n$ and $F_n(R,z) = F_n(R,-z)$ due to periodic boundary conditions and symmetric with respect to $p \to 1 - p$.
[22] Since it is possible to have $n_p = 1$ and $n_n = 0$, W_1 is not invariant with respect to $p \to 1 - p$ and we do not have the symmetry $F_1(R,z) = F_1(R,-z)$.