Liouville Theory as a Model for Prelocalized States in Disordered Conductors

Ian I. Kogan,[1] C. Mudry,[2] and A. M. Tsvelik[1]
[1]*Department of Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, United Kingdom*
[2]*Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139*
(Received 29 February 1996)

It is established that the distribution of the zero energy eigenfunctions of (2 + 1)-dimensional Dirac electrons in a random gauge potential is described by the Liouville model. This model has a line of critical points parametrized by the strength of disorder and the scaling dimensions of the inverse participation ratios coincide with the dimensions obtained in the conventional localization theory. From this fact we conclude that the renormalization group trajectory of the latter theory lies in the vicinity of the line of critical points of the Liouville model. [S0031-9007(96)00728-4]

PACS numbers: 71.23.An, 05.45.+b, 03.65.Sq, 05.60.+w

Among the problems of localization theory there is one which, until recently, had attracted less attention than it deserves. This is the problem of spatial correlations of wave functions at distances much smaller than the localization length L_c (see Refs. [1,2]). This problem is well understood only for extended states, i.e., in the limit of small wave function amplitudes $t = |\psi(\mathbf{x})|^2$. Since extended states explore the entire sample, one can neglect their spatial variations and treat the Hamiltonian as a random matrix. The distribution function $P(t)$ derived by the methods of random matrix theory depends only on the global symmetry of the random ensemble and has an approximately Gaussian form (the Porter-Thomas distribution; see, for example, [3]). This approach fails for larger t's since the tails of the distribution function are determined by rare spatially inhomogeneous configurations with high local amplitudes. The first calculation for the asymptotic form of $P(t)$ in small two-dimensional samples ($L \ll L_c$) was performed by Altshuler, Kravtsov, and Lerner [2] using the renormalization group and replicas. The nonperturbative approach based on the supersymmetric σ model had not been used in this context until Muzykantskii and Khmelnitskii [4] pointed out that in order to describe exceptional events most affected by the disorder one should look for a saddle point of the supersymmetric σ model. This idea has then been exploited by Fal'ko and Efetov [5] who have derived a reduced σ model adapted to the studies of properties of a single quantum state in the discrete spectrum of a confined system and found that the reduced σ model has a nontrivial vacuum. They have calculated $P(t)$ in two loops (the saddle point approximation with Gaussian fluctuations around it).

Below we discuss only two-dimensional systems. In two dimensions the conventional localization theory of unitary ensemble represents a special case since the first loop localization correction to the conductivity vanishes and the localization length is therefore very large: $L_c \sim \exp(G_0^2)$, where G_0 is the bare conductance (we assume that $G_0 \gg 1$). Meanwhile the behavior of wave functions becomes nontrivial at much smaller length scales $L > \exp(G_0)$. In this case the results of Altshuler *et al.* and Fal'ko and Efetov suggest that the renormalization flow for $P(t)$, yet eventually turning to the strong coupling (localization), spends a lot of "time" in the vicinity of some critical line where the asymptotics of $P(t)$ is given by the log-normal distribution. It is difficult, however, to identify this critical line within the replica approach. The supersymmetric saddle point calculations provide a better insight since they give the Liouville equation as the σ model's saddle point condition.

In this Letter we describe a model of disorder where the line of critical points discovered in Refs. [2,5] is stable. This is the theory of (2 + 1)-dimensional Dirac fermions in a random gauge potential (FRGP model). We show that the distribution function of the prelocalized states in this model is described by the Liouville field theory (LFT). This theory was introduced by Polyakov [6] in the context of string theory and has been extensively studied. We are going to use this accumulated knowledge for the theory of localization. One important property of LFT is that it does have a stable line of critical points which seems to describe the prelocalized states in the conventional theory of localization. As we shall show later, this line is parametrized by the strength of disorder.

Until the present time only two critical disordered systems have been studied: the model of a half-filled Landau level (see, for example, [7], and references therein) and the model of (2 + 1)-dimensional Dirac fermions in a random gauge potential (FRGP model) [8–11]. The latter model has an unbounded energy spectrum, but the spectrum of $\hat{H}^2$ is bounded from below ($E_n^2 \geq 0$). Since the conductivity of the FRGP model at zero frequency is finite and the wave functions with $E = 0$ have multifractal properties (see Ludwig *et al.* [8]), we suggest that the mobility edge of this model coincides with the boundary of the spectrum generating operator $\hat{H}^2$. This conjecture does not generally hold in more complex critical disordered systems (e.g., 2D symplectic and 3D).

In this Letter we study the statistics of $E = 0$ wave functions of the Abelian FRGP model in two dimensions [12]. In the gauge where $A_\mu = \epsilon_{\mu\nu}\partial_\nu\Phi$ the Dirac equation has one obvious solution with $E = 0$, which we write down in the normalized form

$$\psi_\sigma(\mathbf{x}) = (1 \pm \sigma^3)_{\sigma,\sigma'} \frac{e^{-\sigma'\Phi(\mathbf{x})}}{[\int d^2x' e^{-2\Phi(\mathbf{x}')}]^{1/2}} . \quad (1)$$

This solution is unique on a closed sphere if the total flux is less than 2. For greater fluxes the index theorem predicts the existence of other solutions with $E = 0$ [13].

We shall consider $\nabla\Phi$ as a random variable with the Gaussian distribution

$$P[\Phi] = Z_0^{-1} \exp\left\{-\frac{1}{4\pi b^2} \int d^2x [\nabla\Phi(\mathbf{x})]^2\right\}. \quad (2)$$

Without loss of generality we can choose $\sigma = 1$ in Eq. (1) and study the moments of the distribution function of the first component of the wave function defined by

$$G(1,\ldots,N) = \int D\Phi P[\Phi] \psi_1^2(\mathbf{x}_1)\cdots\psi_1^2(\mathbf{x}_N) . \quad (3)$$

Notice that all these quantities are invariant under the shift $\Phi \to \Phi + c$ by an arbitrary real constant c. The N-point moment of *normalizable* wave function squares can always be rewritten as follows:

$$G(1,\ldots,N) = \int_0^\infty \frac{d\mu \exp[-\alpha\mu]\mu^{N-1}}{(N-1)!} \times \int D\Phi \prod_{i=1}^N e^{-2\Phi(\mathbf{x}_i)} e^{-S_\mu}, \quad (4)$$

where the action S_μ is given by

$$S_\mu = \int d^2x \left[\frac{1}{4\pi b^2}(\nabla\Phi)^2 + \mu e^{-2\Phi}\right], \quad (5)$$

and the exponent $\exp[-\alpha\mu]$ is introduced to make the integral over μ convergent. Thus the multipoint moment (3) is now expressed in terms of the *reducible* multipoint correlation function of LFT.

Now we shall recall several facts about LFT that we are going to use. They can be found, for example, in Refs. [14,15]. To conform to the notations accepted among the field theorists, we rescale the field $\Phi = -b\phi$. We also impose the following boundary conditions on the Liouville field:

$$\phi(\mathbf{x}) = -Q \ln|\mathbf{x}|^2, \qquad |\mathbf{x}| \longrightarrow \infty, \qquad Q = b + \frac{1}{b} . \quad (6)$$

Thus we are considering only ground state wave functions that decay algebraically at infinity. These are the *prelocalized states*. If boundary conditions are not chosen properly, the correlation functions of LFT vanish. In the semiclassical saddle point calculations [5,14,15], this feature emerges as a condition for the existence of the saddle point. The boundary condition (6) can be formulated as a condition that the total flux through the system is equal to Qb. Uniqueness of the wave functions (1) is not violated provided $Qb < 2$ [16].

There are different ways to implement the boundary condition (6). One can consider the LFT on a large disk Γ of radius $R \to \infty$ and add to the local LFT action (5) a boundary term:

$$S_{\mu,Q} = \frac{1}{4\pi} \int_\Gamma d^2x [(\nabla\phi)^2 + 4\pi\mu e^{2b\phi}] + \frac{Q}{\pi R} \int_{\partial\Gamma} \phi \, dl . \quad (7)$$

It is more convenient, however, to contract the boundary into the point, thereby transforming the disk into a sphere. The boundary condition is then equivalent to the insertion of the Liouville exponential $e^{-2Q\phi(R)}$ ($R \to \infty$) into all correlation functions [17]. For $\mu = 0$ one can express correlation functions of the fields $e^{2\alpha\phi}$ in the LFT with boundary conditions in terms of correlation functions of the conventional Gaussian model

$$\left\langle \prod_i e^{2\alpha_i\phi(x_i)} \right\rangle_Q = \prod_{i<j} |x_i - x_j|^{-4\alpha_i\alpha_j} \times \delta_\epsilon\left(\sum_i \alpha_i - Q\right) R^{-2Q^2},$$

$$\delta_\epsilon(x) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} , \quad (8)$$

where ϵ is a small parameter introduced for regularization and the subscript Q means that the function on the left-hand side is the correlation function calculated with action (7). Provided the *neutrality condition* $\sum_i \alpha_i = Q$ is satisfied, all correlation functions are proportional to the same factor R^{-2Q^2}. We shall see that this factor is compensated by the integral over μ. Calculating two-point correlation functions in LFT, we conclude that the conformal dimension of the Liouville exponential is

$$\Delta(e^{2\alpha\phi}) = \alpha(Q - \alpha) , \quad (9)$$

and this is true for $\mu \neq 0$ also [18–20]. Note that Q has been chosen such that the exponential operator in the Liouville action is marginal, i.e., $\Delta(e^{2b\phi}) = 1$, thus preserving the criticality of the theory.

It follows immediately from Eq. (9) that the conformal dimensions $\Delta(q)$ (q real) of the composite operators $:\psi_1^{2q}(\mathbf{x}): \sim \exp[2qb\phi(\mathbf{x})]$ (the sign $:\cdots:$ denotes normal ordering) are equal to

$$\Delta(q) = q(1 + b^2 - b^2 q). \quad (10)$$

These dimensions depend on the continuous parameter b which represents the disorder strength. The LFT model remains critical for any values of b with the central charge given by

$$C = 1 + 6Q^2 = 1 + 6(b + 1/b)^2. \quad (11)$$

However, at $b^{-2} = M$, where $M = 1, 2, \ldots,$ the theory possesses an additional hidden SL(2, **R**) symmetry (see, for example, [14]). It is also known [15] that three point correlation functions have resonances at these values of b.

Equation (10) reproduces the dimensions obtained for FRGP in Ref. [8] by the replica trick and also the dimensions obtained for the conventional localization theory [2,5] (in the notation of Ref. [5] $b^{-2} = 8\pi^2\beta\nu D$). Needless to say that in the conventional localization theory the parameter b undergoes renormalization towards strong coupling and therefore the described equivalence holds only in the crossover regime. As we have mentioned earlier, an extended crossover region exists only for the unitary ensemble ($\beta = 2$).

An important application of (10) is to the calculation of the scaling with respect to R of the inverse participation ratios

$$\left\langle \int_\Gamma d^2x \psi_1^{2q}(\mathbf{x}) \right\rangle \propto R^{-\tau(q)}, \quad \tau(q) = 2(1 - b^2 q)(q - 1). \tag{12}$$

The applicability of this scaling for large q has been discussed in Refs. [5,10,21]. It was established that $\tau(q)$ must be a monotonously increasing function of q. This means that Eq. (12) is at most valid for $q \le (1 + b^{-2})/2$. This condition selects operators $\exp(2qb\phi)$ with $qb \le (b + 1/b)/2 = Q/2$. Let us note that in the weak disorder limit $b \ll 1$, these exponents can still be used for the description of very high participation ratios.

This selection resembles the following well-known fact from the LFT. Namely, there is a difference between operators $\exp(2\alpha\phi)$ with $\alpha < Q/2$ and $\alpha > Q/2$ [14]: Only the first ones (so-called *microscopic* operators) correspond to local states whereas the latter (so-called *macroscopic* operators) create finite holes on the random surface. In LFT the field $\exp(2b\phi)$ is interpreted as a metric of a two-dimensional surface. In the semiclassical limit $b \ll 1$ correlation functions of microscopic operators can be obtained by the saddle point approximation. The solution of classical equations of motion describes a surface spanned on a disk with radius $R \to \infty$. This surface has constant negative curvature metric with spikes (integrable power law singularities of the metric field $g_{z\bar{z}} \sim |z|^{-\eta}, \eta < 2$) at the points of insertion of operators with $\alpha < Q/2$ (for $\alpha = Q/2$ we have punctures, i.e., $|z|^{-2}$ singularities). For macroscopic operators there is no classical solution corresponding to a surface with a single boundary at infinity which means that each insertion creates a hole of a finite size. Since the exponents with $q > (1 + b^{-2})/2$ correspond to macroscopic operators, it is not correct, in our opinion, to use them for description of higher order participation ratios. It is important to mention, however, that macroscopic operators appear in fusion of microscopic ones.

The authors of Ref. [5] have stressed the importance of the nonuniversal prefactor on the right-hand side of (12) to assess the self-consistency of the scaling analysis. This prefactor is controlled by the dependency on μ of the correlation functions of the LFT. Let us note that above we discussed the results which were independent of μ, so one can put $\mu = 0$. Now we have to study the effects which arise in the full theory with nonzero μ.

It is easy to find the scale (μ) dependence of any correlation function in Liouville theory [18–20]:

$$\begin{aligned}\tilde{G}(1,\ldots,N) &= \left\langle \prod_{i=1}^{N} e^{2\alpha_i \phi(\mathbf{x}_i)} \right\rangle_Q \\ &= (\pi\mu)^{(Q - \sum_{i=1}^{N} \alpha_i)/b} F_{\alpha_1 \cdots \alpha_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N).\end{aligned} \tag{13}$$

Here $F_{\alpha_1 \cdots \alpha_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ must be calculated in the Liouville theory with $\mu = 1$ and the answer for the generic correlation function is unknown. For three- and four-point functions the answer was found in [15,22] for any b. However, for the special case when the factor $(Q - \sum_i \alpha_i)/b$ is an integer, i.e., when the correlation function (13) is proportional to an integer power of μ, all multipoint correlation functions can be obtained explicitly by expanding $\exp(-S_{\mu,Q})$ [cf. (7)] in powers of μ.

Thus unlike the case of conformal dimensions where Eq. (10) holds for any b, the multipoint correlation functions can be easily obtained only for discrete values of the disorder strength $b^2 = 1/M$, $M = 1, 2, \ldots$, for which the expansion over μ contains only *one* nonvanishing term. Indeed when b^{-2} is an integer, the multipoint correlation functions of the Liouville theory can be expressed in terms of correlation functions of the free bosonic field (the Dotsenko-Fateev construction [23]). To see this, we recall that for the correlation function (4) all $\alpha_i = b$. Hence

$$\left(Q - \sum_{i=1}^{N} \alpha_i \right) \Big/ b = 1 - N + 1/b^2. \tag{14}$$

Provided $b^{-2} = M$ with integer $M \ge N - 1$ the neutrality condition is fulfilled with the term $\sim \mu^{1-N+M}$. The result for the N-point function on the infinite plane follows immediately:

$$\tilde{G}(1,\ldots,N) = \epsilon^{-1} R^{-2Q^2} \frac{\mu^{1-N+M}}{(1-N+M)!} \prod_{i<j}^{N} \frac{1}{|z_{ij}|^{4/M}} \int \left(\prod_{l=1}^{1-N+M} d^2\xi_l \right) \left(\prod_{l=1}^{1-N+M} \prod_{i=1}^{N} \frac{1}{|z_i - \xi_l|^{4/M}} \right) \left(\prod_{m<n}^{1-N+M} \frac{1}{|\xi_{mn}|^{4/M}} \right) \tag{15}$$

(here $z, \bar{z} = x \pm iy$ are complex coordinates). The expression for a finite system can be obtained by conformal transformation according to the general rules of conformal theory. Substituting Eq. (15) into Eq. (4) we find that all correlation functions are proportional to the same nonuniversal factor $D = \epsilon^{-1} R^{-2Q^2} \int_0^\infty d\mu \, \mu^{1/b^2} e^{-\alpha\mu}$. We remark that by regularizing the theory with the introduction of $\exp[-\alpha\mu]$, we overestimate the normalization factors of the wave functions. In other words, we restrict the allowed spatial fluctuations of the wave function amplitude from above and thereby eliminate the contributions from localized states to the statistical average in (3).

Finally, we note that as shown in Ref. [21], $b = 1$ is the largest value of the disorder strength for which quenched and annealed averages [with respect to the Gaussian distribution (2)] of the normalization $\int d^2x \exp[-2\Phi(\mathbf{x})]$ agree. On the other hand, $b = 1$ corresponds to the minimum of the Liouville central charge $C_{\min} = 25$. We recall that when $b = 1$ the total flux through the system reaches the value of 2 which corresponds to the (first) change in the ground state degeneracy.

We conclude with a remark about a difference between even and odd M in Eq. (15). For even M, the operator with maximal dimension $\Delta_{\max} = 1/2 + M/4$ is $O_{M/2}(x) \equiv \exp(Mb\phi)$. From Eq. (15) we find

$$\langle O_{M/2}(z_1) O_{M/2}(z_2) \rangle = \frac{D}{|z_{12}|^M} \int \frac{d^2\xi}{|z_1 - \xi|^2 |z_2 - \xi|^2} \approx \frac{4D}{|z_{12}|^{M+2}} \ln(|z_{12}|/a) . \quad (16)$$

Thus the two-point correlation function of this operator contains a logarithm. This does not happen when M is odd. Indeed, the operator with maximal dimension $\Delta_{\max} = (M + 1)^2/4M$ is $O_{(M+1)/2}(x)$ and its two-point correlation function has the conventional form. It may be that, as in other theories with logarithms (see Gurarie [24] and Caux *et al.* [11]), Eq. (16) indicates the presence of an additional operator $C(x)$ in the operator algebra of the theory such that

$$\langle C(z_1) O_{M/2}(z_2) \rangle = -\frac{4D}{|z_{12}|^{M+2}} , \qquad \langle C(z_1) C(z_2) \rangle = 0 . \quad (17)$$

We are not certain, however, that the same interpretation is valid for the Liouville theory where there is no one-to-one correspondence between operators and states and one therefore must be careful formulating the operator expansion. It is also interesting to note that for $b = M = 1$ we have the Liouville theory with central charge $C_{\min} = 25$ which is known to have logarithmic operators with scaling dimensions one [25]. As we discussed earlier, when $b = 1$ the total flux through the system reaches the value of 2 which corresponds to the appearance of a second ground state in our model. The relationship between the existence of marginal logarithmic operator and the change of the ground state degeneracy is an interesting open question.

In more complicated models where the disorder depends on several fields we expect that the corresponding universality classes will be connected with W_n gravities. Another interesting problem is to find what deformation of the Liouville theory leads to localization and thus to reproduce the renormalization group equations obtained by Altshuler *et al.* [2].

We are grateful to J. Chalker, A. Georges, M. Gunn, K. Efetov, M. P. A. Fisher, D. Khveshchenko, V. Kravtsov, D. Khmelnitskii, I. Lerner, A. W. W. Ludwig, A. A. Nersesyan, and especially to V. Fal'ko and E. Fradkin for illuminating discussions, constructive criticism, and interest for the work. C. M. acknowledges financial support from the Swiss National funds.

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