Generalized Dynamic Scaling for Critical Relaxations

B. Zheng
Universität-GH Siegen, D-57068 Siegen, Germany
(Received 15 December 1995)

The dynamic relaxation process for the two dimensional Potts model at criticality starting from an initial state with very high temperature and arbitrary magnetization is investigated with Monte Carlo methods. The results show that there exists universal scaling behavior even in the short-time regime of the dynamic evolution. In order to describe the dependence of the scaling behavior on the initial magnetization, a critical characteristic function is introduced. [S0031-9007(96)00651-5]

PACS numbers: 64.60.Ht, 02.70.Lq, 05.50.+q, 75.40.Gb

For a long time it was believed that no universal behavior would be present in the short-time regime of critical dynamics. However, for the critical relaxation process starting from an initial state with *very high temperature* and *small magnetization*, it was recently argued by Janssen, Schaub, and Schmittmann [1] with renormalization group methods that there exist universality and scaling even *at early times*, which sets in right after a microscopic time scale $t_{\rm mic}$. For the $O(N)$ vector model with dynamics of model A, the authors derived a dynamic scaling relation which is valid up to the macroscopic short-time regime,

$$M^{(k)}(t,\tau,m_0) = b^{-k\beta/\nu} M^{(k)}(b^{-z}t, b^{1/\nu}\tau, b^{x_0}m_0), \quad (1)$$

where $M^{(k)}$ is the kth moment of the magnetization, t is the dynamic evolution time, τ is the reduced temperature, and the parameter b represents the spatial rescaling factor. Besides the well known static critical exponents β, ν, and the dynamic exponent z, a new independent exponent x_0 which is the anomalous dimension of the initial magnetization m_0 is introduced to describe the dependence of the scaling behavior on the initial conditions. Based on the scaling relation it is predicted that at the beginning of the time evolution the magnetization surprisingly undergoes a *critical initial increase*

$$M(t) \sim m_0 t^{\theta}, \quad (2)$$

where the exponent θ is related to the exponent x_0 by $\theta = (x_0 - \beta/\nu)/z$. This makes the effect of m_0 very prominent.

Numerical simulation supports the above predictions for the critical short-time dynamics. The exponent θ for the Ising model was first obtained indirectly through the power law decay of the autocorrelation [2,3]. Recently the initial increase of the magnetization in (2) was observed for the Ising model and the Potts model, and the exponent θ was directly measured [4,5]. The scaling relation and the universality are confirmed [4,6,7]. The investigation of the universal behavior of the short-time dynamics not only enlarges the fundamental knowledge of critical phenomena but also, more interestingly, provides possible new ways to determine all the static exponents as well as the dynamic exponents from the short-time dynamics, either based on the power law behavior of the observables at the beginning of the time evolution [5,8,9], or on the finite size scaling [10,11]. Moreover, the universal behavior of the short-time dynamics is found to be quite general, e.g., in the dynamics beyond model A or at the tricritical point [12–14], in connection with ordering dynamics or damage spreading [15,16] and on the surface critical phenomena [17]. Therefore, thorough understanding of the universality and scaling for the short-time dynamics is urgent and important.

The scaling relation (1) is valid only under the conditions that the initial state is at very high temperature and with *small* initial magnetization. Are there some reasons that the universal behavior emerges only in the critical relaxation starting from such a special initial state? If one believes that the large time correlation length is essential for the universality, the only background one would find is that the initial temperature $T_0 = \infty$ and the initial magnetization $m_0 = 0$ are the fixed points under the renormalization group transformation. Therefore one may wonder whether there exists universal behavior in the critical relaxation process starting from an initial state with very high temperature but initial magnetization $m_0 \simeq 1$, since $m_0 = 1$ also corresponds to a fixed point. Actually the critical relaxation process with $m_0 = 1$ for the Ising model and the Potts model have been investigated with Monte Carlo simulations [8,11,18,19]. The results show that universality and scaling appear to be valid also in the early stage of the time evolution.

In this Letter we are more ambitious. We study whether there exists universal scaling behavior in the short-time regime of the critical relaxation from an initial state with very high temperature and *arbitrary magnetization.* If the large time correlation length plays an essential role for the universality in the critical dynamics as it was pointed out above, the presentation of the universal behavior should not rely on from what initial conditions the critical relaxation starts, and only the scaling behavior of the initial conditions should be considered very carefully. Since the arbitrarily valued initial magnetization is no more around a fixed point, a critical exponent x_0 is not sufficient to describe the scaling behavior of the initial magnetization. In the language of the renormalization group method, the effective dimension of the initial magnetization will in general depend on the initial magnetization itself. In order to describe this phenomena, we introduce a *critical characteristic function* rather than a critical exponent. For the kth moment of the magnetization, the generalized scaling relation may be written as

$$M^{(k)}(t, \tau, L, m_0) = b^{-k\beta/\nu} M^{(k)}(b^{-z} t, b^{1/\nu} \tau, b^{-1} L, \chi(b, m_0)) . \quad (3)$$

For the convenience of later discussion, finite systems have been considered here and L is the lattice size. The scaling behavior of the initial magnetization m_0 is specified by the critical characteristic function $\chi(b, m_0)$, which in the limit $m_0 \rightarrow 0$ tends to the simple form $b^{x_0} m_0$, but is in general different. Such a generalized scaling form is in a similar spirit as that in the correction to the scaling, where nonlinear effects of an off-fixed point are considered. Here only our initial magnetization is a relevant operator and can be far away from the fixed point. The ansatz that the exponents β, ν, and z do not depend on the initial magnetization m_0 is based on the assumption that the initial conditions should not enter the renormalization of the critical system in equilibrium and near equilibrium. In other words, if there is a scaling form in the short-time regime of the dynamic relaxation process, the scaling form should smoothly cross over to that in the long-time regime. This greatly simplifies the short-time behavior of the critical dynamics. In the neighborhood of $m_0 = 1$ the critical characteristic function $\chi(b, m_0)$ may also be characterized by a critical exponent. One can also realize that in the limit of $b = 0$, $\chi(b, m_0) \rightarrow 0$, and when b approaches infinity, $\chi(b, m_0) \rightarrow 1$. In this Letter we are interested in the more general case; i.e., m_0 is between 0 and 1, and b is a reasonable finite number. The limiting cases will be discussed in detail elsewhere.

Here we stress that the scaling relation in (3) is not trivial, even though the critical characteristic function $\chi(b, m_0)$ looks not so "*simple*" as a critical exponent. The scaling relation relates the time evolution of the observables with different initial magnetizations to each other and represents the *self-similarity* of the dynamic systems. *All the physical observables as functions of t, τ, and L should be described by the same critical characteristic function* $\chi(b, m_0)$. Besides this, the physical observables are universal functions of the variables t, τ, and L up to a nonuniversal scaling constant, and the dependence of the observables as well as $\chi(b, m_0)$ on m_0 is expected to be universal up to a rescaling of the initial magnetization m_0.

As a concrete example, we consider the two dimensional Potts model, for which a quite accurate dynamic exponent z has been obtained from the universal behavior of the short-time dynamics [5,7]. The Hamiltonian for the q state Potts model is given by

$$H = K \sum_{\langle ij \rangle} \delta_{\sigma_i \sigma_j}, \quad \sigma_i = 1, \ldots, q , \quad (4)$$

where $\langle ij \rangle$ represents nearest neighbors. In our notation the inverse temperature is already absorbed into the coupling K. It is known that the critical points locate at $K_c = \log(1 + \sqrt{q})$. In this paper the three state ($q = 3$) case will be investigated. In generating the random initial configurations, we have sharply prepared the initial magnetization in order to avoid extra finite size effects from the fluctuation of m_0 in the finite systems. After the preparation of the initial configurations the system is released to evolve according to the heat-bath algorithm.

For simplicity, we take $\tau = 0$, and therefore the exponent ν will not enter the calculation. The exact value of the static exponents $\beta/\nu = 2/15$ and the dynamic exponent $z = 2.196$ obtained from the power law decay of the autocorrelations [5,7] will be taken as input. To verify the scaling relation (3) and determine the critical characteristic function $\chi(b, m_0)$, we perform the simulation for a pair of lattice sizes L_1 and L_2 and measure the time evolution of the magnetization and the second moment defined as

$$M^{(k)}(t, m_0) = \left\langle \left[\frac{3}{2N} \sum_i [\delta_{\sigma_i(t),1} - \tfrac{1}{3}] \right]^k \right\rangle, \quad k = 1, 2 , \quad (5)$$

where the average is taken over independent initial configurations and the random forces. In order to reduce extra errors from $M^{(k)}(t, 1)$, especially when m_0 becomes bigger, we introduce a magnetization difference

$$M_d(t, m_0) = M^{(1)}(t, 1) - M^{(1)}(t, m_0) \quad (6)$$

and a Binder-type cumulant

$$U_d(t, m_0) = \frac{M^{(2)}(t, 1) - M^{(2)}(t, m_0)}{[M_d(t, m_0)]^2} . \quad (7)$$

From the scaling collapse of $M_d(t, m_0)$ or $U_d(t, m_0)$ for two lattices with suitable initial magnetizations, we can estimate the values of the function $\chi(b, m_0)$ at $b = L_2/L_1$ for different m_0.

In Fig. 1 the scaling plot for the magnetization is displayed for the lattice $L_2 = 144$ with initial magnetization $m_{02} = 0.14$ and the lattice $L_1 = 72$ with suitably corresponding initial magnetizations m_{01}. The stars represent the time evolution of the magnetization of the lattice $L_2 = 144$. The crosses are the same data but rescaled in time and multiplied by an overall factor $2^{\beta/\nu}$. If one can find a m_{01} for lattice $L_1 = 72$ such that its time evolution of the magnetization fits to the crosses, the scaling is valid, and from the scaling relation (3) one gets $\chi(2, m_{02}) = m_{01}$. Practically we have performed the

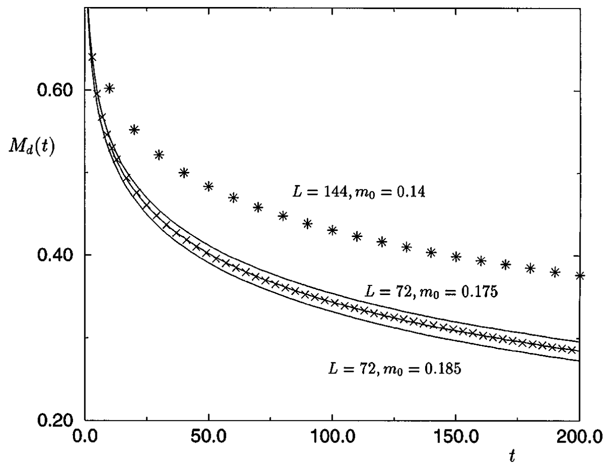

FIG. 1. The scaling plot for the magnetization with $(L_1, L_2) = (72, 144)$.

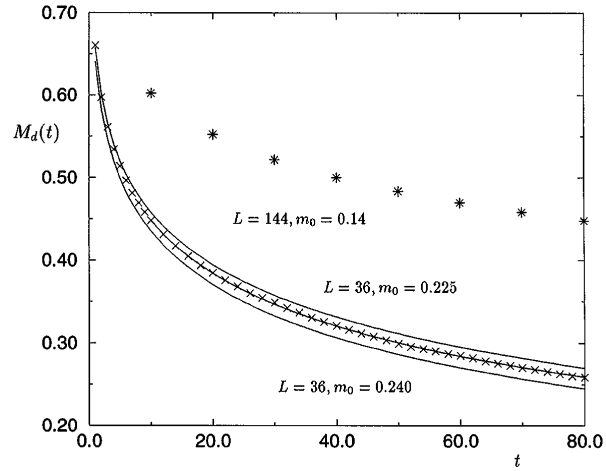

FIG. 2. The scaling plot for the magnetization with $(L_1, L_2) = (36, 144)$.

simulation for $L_1 = 72$ with two different initial magnetizations $m_0 = 0.175$ and $m_0 = 0.185$, for which the magnetizations have been plotted by the lower and upper solid lines in Fig. 1. By linear extrapolation, we obtain the time evolution of the magnetization with initial values between these two values. Then we can estimate m_{01}. The solid line laying on the crosses in Fig. 1 is the time evolution of the magnetization for $L = 72$ with an initial magnetization $m_0 = 0.1800(4)$ which has the best fit to the crosses, i.e., $\chi(2, 0.14) = 0.1800(4)$. From the previous numerical simulations for the short-time dynamics [4,5,7], we know that the microscopic time scale t_{mic} is negligibly small for the heat-bath algorithm, at least for the measurement of the dynamic exponent θ or the exponent x_0. In our calculation we have carried out the fitting procedure in a time interval of [10,200] in the time scale of lattice $L = 72$. From Fig. 1 one can also see clearly that the scaling relation is valid starting from the very early stage of the time evolution.

In Fig. 2 a scaling plot is shown for the lattice $L_2 = 144$ with $m_{02} = 0.14$ and $L_1 = 36$. From a fitting in a time interval of [10,80] we obtained $\chi(4, 0.14) = 0.2328(9)$. In a similar way, $\chi(b, m_0)$ may be independently obtained from the scaling collapse of the Binder-type cumulant $U_d(t, m_0)$. Altogether we have performed the simulation for $m_{02} =$ 0.14, 0.22, 0.30, 0.40, 0.50, 0.60, 0.70, 0.80, and 1.00. In Table I, the results for two typical values $m_{02} = 0.14$ and 0.40 are given. We can see that $\chi(b, m_0)$ estimated from both $M_d(t, m_0)$ and $U_d(t, m_0)$ are very consistent. In order to see the finite size effect, we have also performed the calculation for other pairs of lattices as, e.g., (36, 72). In Table I, we can see that the finite size effect for the lattice pair $(L_1, L_2) = (36, 72)$ is already quite small. In the simulations, the statistics of $L = 36$, $L = 72$, and $L = 144$ are, respectively, 80 000, 40 000, and 8000. In Table I, errors are estimated by dividing the data into four groups. In principle the measurements from the magnetization are more reliable than those from the Binder-type cumulants where higher moments are involved, at least when the static exponent β/ν is known. The magnetization is self-averaging, but the Binder-type cumulant is not. When m_0 is getting bigger, $U_d(t, m_0)$ is more fluctuating.

In order to get a more direct understanding of the full critical characteristic function $\chi(b, m_0)$, we define an effective dimension $x(b, m_0)$ of the initial magnetization m_0 by $\chi(b, m_0) = b^{x(b,m_0)} m_0$. By the definition naturally $x(b, 0) = x_0$. From the values of $\chi(b, m_0)$, we can calculate $x(b, m_0)$. Taking the results from the magnetization difference with the lattice pair $(L_1, L_2) = (36, 144)$ and $(72, 144)$, the corresponding effective dimensions $x(b, m_0)$ are plotted in Fig. 3. It clearly shows that $\chi(b, m_0)$ is a nontrivial function. For example, the values $x(2, 0.14) = 0.363(3)$ and $x(2, 0.40) = 0.408(2)$ apparently differ from $x_0 = 0.298(6)$ [7]. When m_0 varies from zero to one, the effective dimension $x(b, m_0)$ first increases and then decreases to zero. However, when m_0 [or $\chi(b, m_0)$] is approaching m_0 [or $\chi(b, m_0)$] $= 1$ it is not the best choice to determine $x(b, m_0)$ directly in the way discussed in this Letter since the dependence of the physical observables on m_0 becomes weaker and we face big statistical fluctuations.

Finally, let us have some more understanding of the dynamic system discussed above. Supposing m_0 is also a "coupling" of the system, in the "K-m_0" plane of the couplings there exists a critical line $K = K_c$ in the sense that the time correlation is divergent. In the neighborhood of this critical line, the time correlation length depends only on the coupling K, and its scaling behavior is characterized by the exponent νz. If the critical line is a line of the fixed points, the exponent ν together with z and β is sufficient to describe the critical dynamic system. However, in our case the critical line is *not a line of the fixed points.* Therefore a critical characteristic function

TABLE I. $\chi(b, m_0)$ measured from the magnetization and the Binder-type cumulant.

	m_0	0.14		0.40	
(L_1, L_2)		M_d	U_d	M_d	U_d
(36, 72)	$\chi(2, m_0)$	0.1807(05)	0.1804(08)	0.5298(06)	0.5300(36)
(72, 144)		0.1800(04)	0.1798(04)	0.5307(06)	0.5302(45)
(36, 144)	$\chi(4, m_0)$	0.2328(09)	0.2324(11)	0.6918(28)	0.6910(78)

should in general be introduced to describe the scaling behavior. It is actually interesting to see what happens for a system in the equilibrium where a critical line exists but the line is not a line of fixed points.

In conclusion, we have numerically simulated the universal short-time behavior of the dynamic relaxation process for the two dimensional critical Potts model starting from an initial state with very high temperature and arbitrary magnetization. The results show that the traditional scaling relation should be generalized. A critical characteristic function is introduced to describe the scaling behavior of the initial magnetization. We demonstrate how to determine numerically the critical characteristic function. The study of the short-time dynamics is not only conceptually interesting but also practically important since it is possible to obtain the static exponents β, ν, and the dynamic exponent z of the critical systems far before the dynamic process reaches the equilibrium. It is challenging to derive analytically the generalized scaling relation in (3) as well as the critical characteristic function $\chi(b, m_0)$. The application to the *dynamic* field theory, e.g., the stochastic quantization of the field theory, is attractive since such a knowledge would be important for the numerical simulation of the lattice gauge theory.

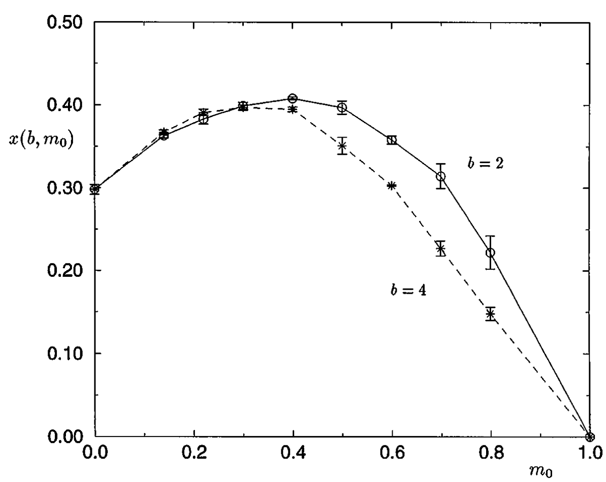

FIG. 3. The effective dimension obtained from the magnetization with $(L_1, L_2) = (72, 144)$ and $(36, 144)$. Circles are of $x(2, m_0)$, and stars are of $x(4, m_0)$. The lines are drawn to guide the eyes.

The author would like to thank L. Schülke, S. Marculescu, and K. Okano for very helpful discussions and K. Untch for maintaining the workstations.

[1] H. K. Janssen, B. Schaub, and B. Schmittmann, Z. Phys. B **73**, 539 (1989).
[2] D. A. Huse, Phys. Rev. B **40**, 304 (1989).
[3] K. Humayun and A. J. Bray, J. Phys. A **24**, 1915 (1991).
[4] Z. B. Li, U. Ritschel, and B. Zheng, J. Phys. A **27**, L837 (1994).
[5] L. Schülke and B. Zheng, Phys. Lett. A **204**, 295 (1995).
[6] N. Menyhárd, J. Phys. A **27**, 663 (1994).
[7] K. Okano, L. Schülke, K. Yamagishi, and B. Zheng, "Universality and Scaling in Short-time Critical Dynamics," University of Siegen, 1995 (to be published).
[8] L. Schülke and B. Zheng, "Determination of the Critical Point and Exponents from Short-time Dynamics," Phys. Lett. A (to be published).
[9] P. Czerner and U. Ritschel, "Universal Short-time Dynamics in the Kosterlitz-Thouless Phase," University of Essen, 1995 (to be published).
[10] Z. B. Li, L. Schülke, and B. Zheng, Phys. Rev. Lett. **74**, 3396 (1995).
[11] Z. B. Li, L. Schülke, and B. Zheng, Phys. Rev. E **53**, 2940 (1996).
[12] H. K. Janssen, *From Phase Transition to Chaos*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél, Topics in Modern Statistical Physics (World Scientific, Singapore, 1992).
[13] K. Oerding and H. K. Janssen, J. Phys. A **26**, 3369 (1993); **26**, 5295 (1993).
[14] K. Oerding and H. K. Janssen, J. Phys. A **27**, 715 (1994).
[15] A. J. Bray, Adv. Phys. **43**, 357 (1994).
[16] P. Grassberger, Physica (Amsterdam) A **214**, 547 (1995).
[17] U. Ritschel and P. Czerner, Phys. Rev. Lett. **75**, 3882 (1995).
[18] D. Stauffer, Physica (Amsterdam) **186A**, 197 (1992).
[19] C. Münkel, D. W. Heermann, J. Adler, M. Gofman, and D. Stauffer, Physica (Amsterdam) **193A**, 540 (1993).