Transitions to Bubbling of Chaotic Systems

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Certain dynamical systems exhibit a phenomenon called bubbling, whereby small perturbations induce intermittent bursting. In this Letter we show that, as a parameter is varied through a critical value, the transition to bubbling can be "hard" (the bursts appear abruptly with large amplitude) or "soft" (the maximum burst amplitude increases continuously from zero), and that the presence or absence of symmetry in the unperturbed system has a fundamental effect on these transitions. These results are confirmed by numerical and physical experiments. [S0031-9007(96)01852-2]

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Dynamical systems that possess an invariant surface embedded in their phase space [1] display unusual dynamical properties and are of interest because of their potential frequent occurrence in applications. Examples include extended systems with spatial symmetries (e.g., Rayleigh-Bérnard convection in a symmetric cell [2], reaction-diffusion systems [2,3], etc.), as well as synchronized chaotic oscillators [4]. The latter are of interest, for example, in various problems of communications [5] and optics [6]. Among the dynamical behaviors characteristic of systems with invariant phase space surfaces are on-off intermittency [7], riddled basins of attraction [8], and bubbling [9–12]. In bubbling there is a chaotic set on the invariant surface which is stable (i.e., an attractor) in the sense that it attracts *typical* orbits near the surface, but, is also unstable in the sense that there are unstable periodic orbits embedded in the chaotic set which are transversely repelling (i.e., they have a positive Lyapunov exponent for perturbations transverse to the invariant surface). This situation leads to a surprising effect: Small changes in the dynamical system that destroy its invariant surface ("mismatch") [13] or noise result in a continual sequence of intermittent bursts from the invariant manifold, *no matter how small* the mismatch or noise. The mean frequency of the bursts, however, approaches zero as the amplitude of the noise and system mismatch approach zero.

In this Letter we investigate the transition to bubbling as a system parameter is varied. Our considerations are limited to the effect of mismatch in the absence of noise. The critical value of the system parameter at which bubbling first occurs typically corresponds to the value at which one of the periodic orbits embedded in the chaotic set first becomes repelling in a direction transverse to the invariant surface [14]. Our principal result is that this bifurcation comes in four basic varieties, each of which yields distinct behaviors that we describe and demonstrate numerically and experimentally.

For specificity, consider the case of two coupled oscillators,

$$\dot{\mathbf{u}} = \mathbf{F}_1(\mathbf{u}) + k_1\mathbf{f}(\mathbf{u} - \mathbf{v}), \tag{1a}$$

$$\dot{\mathbf{v}} = \mathbf{F}_2(\mathbf{v}) + k_2\mathbf{f}(\mathbf{v} - \mathbf{u}), \tag{1b}$$

where $\mathbf{f}$ and $\mathbf{F}_{1,2}$ are smooth functions, $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, and k_1 and k_2 are "coupling constants." For the "matched" case $\mathbf{F}_1(\mathbf{w}) = \mathbf{F}_2(\mathbf{w})$, the synchronized state $\mathbf{u} = \mathbf{v}$ represents an invariant surface embedded in the full $(\mathbf{u}, \mathbf{v})$ phase space. Note that Eqs. (1a) and (1b) have a symmetry when $k_1 = k_2$: They are unchanged when $\mathbf{u}$ and $\mathbf{v}$ are interchanged. Thus, we say that there is "symmetric coupling" when $k_1 = k_2$, while when $k_1 \neq k_2$ we say that the coupling is asymmetric. We will show below that the properties of the transition to bubbling differ for the symmetric and asymmetric coupling cases. Note that extended systems with a spatial symmetry have the same generic properties as coupled oscillators with symmetric coupling [2].

The universality of the transition to bubbling implies that very simple models that incorporate the essential features responsible for these phenomena can be used to extract general results. In this spirit we introduce the following model system:

$$x_{n+1} = 2x_n \bmod 1, \tag{2a}$$

$$y_{n+1} = [\lambda(x_n, p)y_n + \epsilon y_n^\sigma + q\cos(2\pi x_n)]_*, \tag{2b}$$

where $\epsilon = \pm 1$ and $\lambda(x, p) = 1 + p - [1 - \cos(2\pi x)]$. The function $[.]_*$ is defined by $[\xi]_* = \xi$ if $|\xi| \leq 1$ and $[\xi]_* = \xi - 1.5\,\mathrm{sgn}(\xi)$ if $|\xi| > 1$ and provides a simple confining nonlinearity that prevents orbits from running off toward $|y| = \infty$. As our subsequent analysis shows, our derived scalings are dictated by behavior near $y = 0$, and are therefore unaffected by the form of the confining nonlinearity. [In terms of Eqs. (1), we can think of x and y in (2) as modeling the dynamics of $\mathbf{x} = (\mathbf{u} + \mathbf{v})/2$ along the invariant surface and $\mathbf{y} = (\mathbf{u} - \mathbf{v})/2$ transverse to this surface, respectively.] The term $q\cos(2\pi x)$ represents a small mismatch [e.g., it models the difference $\mathbf{F}_1 - \mathbf{F}_2$ in Eqs. (1)].

In the absence of mismatch (i.e., for $q = 0$), Eqs. (2a) and (2b) have an invariant line $y = 0$ on which there is a chaotic invariant set generated by the $2x \bmod 1$ map, Eq. (2a). The stability of this line is governed by the factor $\lambda(x,p)$, where p is the bifurcation parameter (e.g., p might characterize the strength of the coupling in (1a) and (1b) where increasing coupling corresponds to decreasing p). Since $\lambda(x,p)$ is maximum at $x = 0$, and since $x = 0, y = 0$ is a period one orbit of the map (2) for $q = 0$, we see that this period one orbit is the first periodic orbit to become transversely unstable, and it does so as p increases through zero. Thus $p = 0$ is the critical parameter value at the transition to bubbling.

The term ϵy^σ in Eq. (2b) represents the lowest order y nonlinearity of the system. The difference between the symmetric coupling case ($k_1 = k_2$) and the asymmetric coupling case ($k_1 \neq k_2$) is reflected in the model (2a) and (2b) by the value of the exponent σ. In particular, Eqs. (2a) and (2b) must be invariant under the symmetry transformation $y \rightarrow -y$ for symmetric coupling. Thus for the symmetric case a y^2 nonlinearity is ruled out, and in the absence of any further symmetries or restrictions on the original system, we generically have that $\sigma = 3$ in the symmetric case, while $\sigma = 2$ in the asymmetric case. In general, one might also add small stochastic perturbations to the right hand sides of Eqs. (2a) and (2b) to model the effect of noise. In this Letter we concentrate on the effect of small mismatch ($1 \gg |q| > 0$) in the absence of noise. (In a real system with small noise our predicted scalings would apply when $|q|$ exceeds the noise level.)

Figures 1(a)–1(d) show time series generated by Eqs. (2a) and (2b) for the symmetric case $\sigma = 3$. Comparing Figs. 1a and 1b, which are for $\epsilon = 1$, we see that the maximum burst amplitude remains about the same as the parameter $p > 0$ is increased. This contrasts with the case $\epsilon = -1$ [Figs. 1c and 1d] for which we see that the maximum burst amplitude increases as $p > 0$ is increased. In both cases the time between bursts increases as the transition is approached. We call the transition where, as p increases through $p = 0$, the bursts appear abruptly with large amplitude ($\epsilon = +1$), a *hard* bubbling transition; and we call the transition where the burst amplitude increases continuously from zero as p increases through zero ($\epsilon = -1$) a *soft* bubbling transition. Hard and soft transitions, qualitatively similar to Fig. 1, also occur for the case of asymmetric coupling, where $q\epsilon > 0$ corresponds to the hard transition, and $q\epsilon < 0$ corresponds to the soft transition.

There are basic differences between the symmetric and asymmetric cases in terms of how the maximum burst amplitude Δ and the average time τ between bursts scale with the parameters p and q. The differences in scalings for the four basic varieties of bubbling transitions are summarized in Table I (see [15]).

We now give a brief derivation of the results listed in Table I (for more detail see Ref. [15]). Consider first an orbit starting at $(x_0, y_0) = (0, 0)$. Subsequent iterates remain at $x = 0$ and obey $y_{n+1} - y_n = py_n + \epsilon y_n^\sigma + q$.

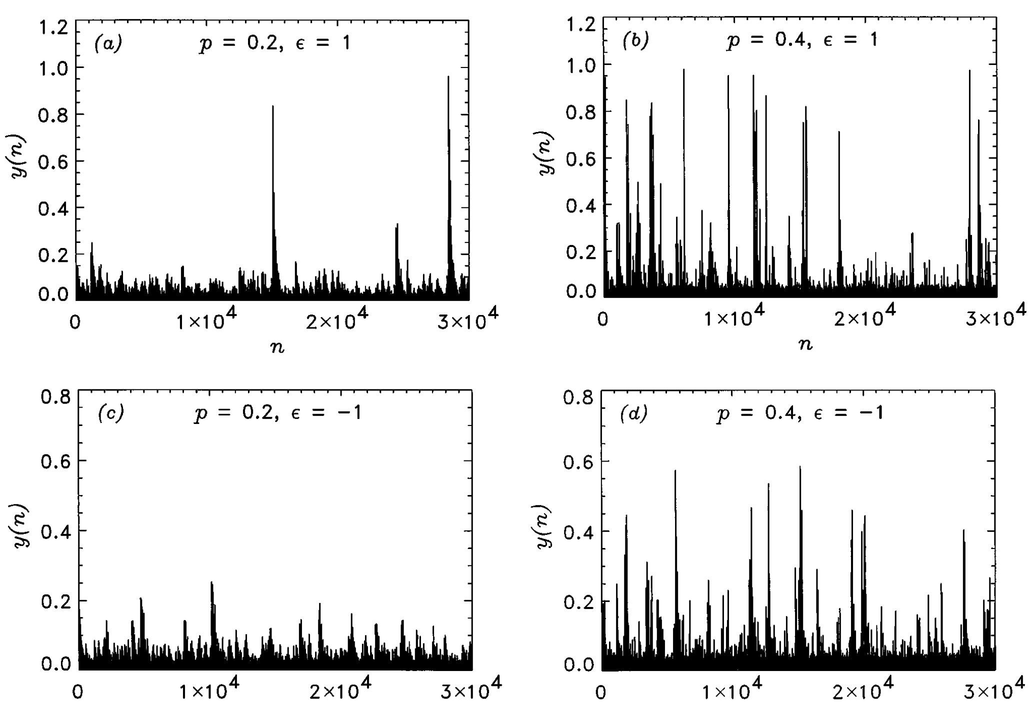

FIG. 1. Examples of the hard (a), (b) and soft (c), (d) transitions using the map model Eqs. (2) with $q = 0.02$ and $\sigma = 3$.

TABLE I. Scaling summary applying for $1 \gg p \gg q^{(1-\sigma^{-1})}$.

	Symmetric coupling ($\sigma = 3$)	Asymmetric coupling ($\sigma = 2$)
Soft transition	$\Delta \sim p^{1/2}$	$\Delta \sim p$
	$\ln\tau \sim h_{\parallel} p^{-1} \ln p/q^{2/3}$	$\ln\tau \sim h_{\parallel} p^{-1} \ln p/q^{1/2}$
Hard transition	$\Delta \sim 0(1)$	$\Delta \sim 0(1)$
	$\ln\tau \sim h_{\parallel} p^{-1} \ln p/q^{2/3}$	$\ln\tau \sim h_{\parallel} p^{-1} \ln p/q^{1/2}$

For small p, q, and y, we can approximate this equation as a differential equation, $dy(n)/dn = py(n) + \epsilon y^{\sigma}(n) + q$, with $y(0) = 0$. It can be shown [15] that most of the time spent in initiating a burst is spent in the region $|y| \lessapprox p^{(\sigma-1)^{-1}}$ in which the nonlinear term y^{σ} is small compared to the linear term py.

For the soft transition, the nonlinearity counteracts the linear growth and limits $|y|$ to $|y| \lessapprox p^{(\sigma-1)^{-1}}$. Thus when $|q|$ is sufficiently small, the maximum burst amplitude is

$$\Delta \sim p^{(\sigma-1)^{-1}} \tag{3}$$

for the soft transition. This result applies for $p\Delta \gg |q|$ (i.e., $p^{\sigma/(\sigma-1)}/|q| \gg 1$). When this condition is not satisfied the approach of Δ to zero as $p \to 0$ implied by (3) is cut off, and a more precise maximum burst amplitude for our model (2a) and (2b) is given by the positive root of

$$p\Delta - \Delta^{\sigma} + q = 0 \tag{3$'$}$$

(assuming $\epsilon = -1$, $q > 0$).

For the hard transition, the nonlinearity accelerates the growth in $|y|$, and when $|y|$ becomes of order $p^{(\sigma-1)^{-1}}$, the orbit rapidly moves off to $|y| \sim 0(1)$. In either case, the time $\bar{n}$ for the initial point $(x_0, y_0) = (0, 0)$ to reach $|y| \sim p^{(\sigma-1)^{-1}}$ can be estimated from the differential equation approximation as

$$\bar{n} \cong \int_0^{p^{(\sigma-1)^{-1}}} dy/(py + |q|) \cong p^{-1} \ln[p^{\sigma/(\sigma-1)}/|q|], \tag{4}$$

for $p^{\sigma/(\sigma-1)}/|q| \gg 1$.

To estimate the average interburst time τ we note that for $p \ll 1$, the factor $\lambda(x, p)$ is less than one (contracting) unless x is very close to 0. Thus in order to initiate a burst, an orbit must land sufficiently near $x = 0$ that it follows the $x = 0$ orbit closely for approximately $\bar{n}$ iterates. Such orbits must land in a region $|x| \lessapprox \Delta x$ where Δx can be estimated in terms of $\bar{n}$, as follows. Since the measure generated by (2a) is uniform in x, we have that $\tau^{-1} \sim \Delta x$. For small initial x_0 the subsequent x values grow exponentially like $x_0 \exp(h_{\parallel} n)$, where $h_{\parallel}$ is the Lyapunov exponent for the critical period orbit in the direction along the invariant surface [$h_{\parallel} = \ln 2$ for Eqs. (2a) and (2b)]. Thus we obtain $\Delta x \exp(h_{\parallel}\bar{n}) \sim 1$ which with (4) yields the estimate [16]

$$\ln\tau \sim h_{\parallel}\bar{n} \sim (h_{\parallel}/p) \ln[p^{\sigma/(\sigma-1)}/|q|]. \tag{5}$$

Eqs. (3) and (5) give the results listed in Table I.

Figure 2 is an example of the scaling of τ with p, for the case of asymmetric coupling ($\sigma = 2$) and a hard transition ($q\epsilon > 0$). The data plotted as diamonds are obtained from numerical experiments on Eqs. (2). The plotted straight line has the slope $2h_{\parallel}$ predicted by Eqs. (4) and (5) and agrees with the data. The scaling of the soft transition maximum burst amplitude showing the effect of small finite q is illustrated by data from results of numerical experiments on Eqs. (2) plotted as diamonds and triangles in Figs. 3(a) and 3(b). Figure 3(a) is for the asymmetric case ($\sigma = 2$), and Fig. 3(b) is for the symmetric case ($\sigma = 3$). In Figs. 3(a) and 3(b) the solid lines are the asymptotic results [Eq. (3)], while the dashed curves include the effect of small finite q [Eq. (3$'$)]. For numerical tests of the scaling of τ with $|q|$ predicted by (5) see [15].

To test our predictions in an experimental setting, we have measured the maximum burst amplitude for two proportional, one-way coupled [$k_2 = 0$ in Eqs. (1)] chaotic electronic circuits as a function of the coupling strength [11] [k_1 in Eqs. (1a) and (1b)]. The layout of an individual circuit is shown schematically in the inset of Fig. 3(c), and the components of the two circuits are matched to within 1%. The dynamics of an individual circuit in the absence of coupling can be described in three-dimensional phase space by $\mathbf{z}_j^T = (V_{1j}, V_{2j}, I_j)$, where $j = m$ or s for the master or slave circuit and it displays a chaotic attractor with one positive Lyapunov exponent.

We observe a transition to bubbling when a current equal to $c(V_{1m} - V_{1s})$ is injected into the "V_1 node" of the slave circuit, where c is the coupling strength. Figure 3(c) shows the dependence of the maximum burst amplitude $\Delta = \|\mathbf{z}_m - \mathbf{z}_s\|_{\max}$ on the coupling strength where it is seen that Δ decreases smoothly, indicating a soft transition

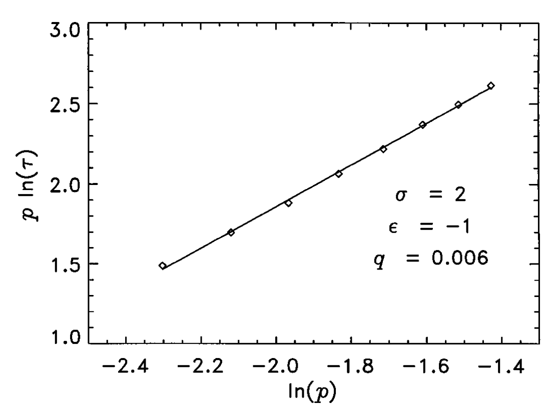

FIG. 2. $p \ln\tau$ versus $\ln p$.

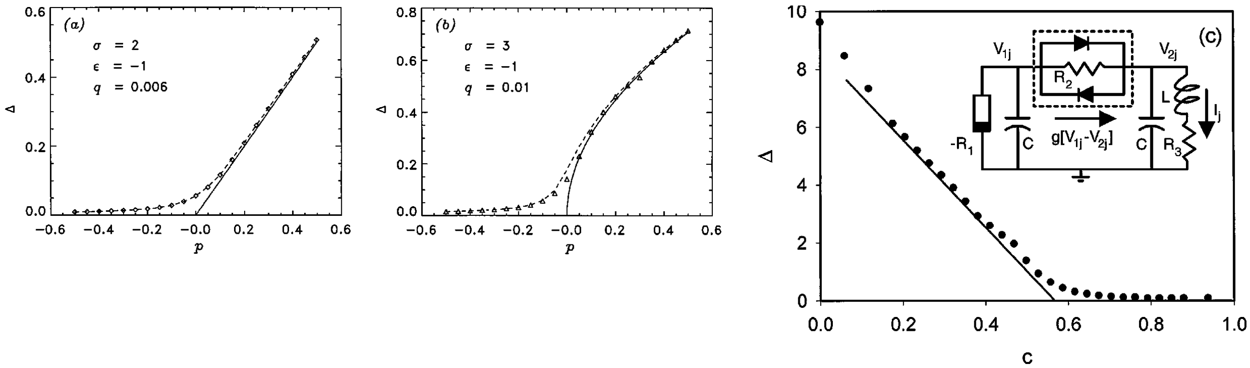

FIG. 3. Maximum burst amplitude Δ versus the bifurcation parameter p for a soft bubbling transition obtained from numerical experiments on Eqs. (2a) and (2b) (a) asymmetric coupling ($\sigma = 2$) (b) asymmetric coupling ($\sigma = 3$). (c) Experimentally measured Δ as a function of coupling strength c. An individual circuit, shown schematically in the inset, consists of a negative resistor $R_1 = 2814\ \Omega$, capacitors $C = 10$ nF, an inductor $L = 55$ mH (dc resistance 353 Ω), a resistor $R_3 = 100\ \Omega$, and a passive nonlinear element (resistor $R_2 = 8{,}067\ \Omega$, diodes type $1N914$, dashed box).

to bubbling. Based on Eq. (3) with $\sigma = 2$, we expect $\Delta \sim (c_b - c)$ where $c = c_b$ is the critical bubbling transition value. It is seen that there is good agreement with the observed points (solid circles) and the straight line for $c_b = 0.56$. Finite noise and mismatch smooth out the transition to bubbling as in the simple model [see Fig. 3(a) and 3(b)].

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