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Bose-Einstein Condensates in Time Dependent Traps

Y. Castin and R. Dum
Ecole Normale Supérieure, Laboratoire Kastler Brossel,. 24 Rue Lhomond, F-75231 Paris Cedex 05, France*
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We present analytical results for the macroscopic wave function of a Bose-Einstein condensate in a time dependent harmonic potential. The evolution of the spatial density is a dilatation, characterized by three scaling factors which allow a classical interpretation of the dynamics. This approach is an efficient tool for the analysis of recent experimental results on the expansion and collective excitation of a condensate. [S0031-9007(96)01919-9]

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Recently the combination of laser cooling and evaporative cooling led to the observation of Bose-Einstein condensation in dilute atomic vapors [1–3]. The favored observation technique has been a time of flight measurement: the trapping potential is rapidly switched off and the spatial distribution of the expanding cloud is monitored. In more recent experiments the condensates were collectively excited by a time modulation of the trapping potential [4,5]. In these experiments the state of the condensate is strongly influenced by atomic interactions, which must therefore be included in a theoretical treatment. Work up to now consisted in the numerical solution of the time dependent nonlinear Schrödinger equation for the macroscopic wave function of the condensate [6]. We present here analytical results which allow a more lucid description of the condensate dynamics and an immediate comparison with experiment. To this end we introduce a quantum scaling transform [7] which is inspired by a model of a classical gas. Applying our results to time of flight measurements of expanding condensates [3] we obtain the scattering length of sodium. For condensates collectively excited by a time modulation of the trapping potential we present an *ab initio* calculation of the observed signal.

For dilute gases at low temperatures the atomic interactions can be modeled by a pseudopotential $g\delta(\vec{r})$, where $g > 0$ is related to the s-wave scattering length a by $g = 4\pi\hbar^2 a/m$ [8]. We describe the trap by an anisotropic time dependent harmonic potential

$$U(\vec{r},t) = \frac{1}{2}\sum_{j=1,2,3} m\omega_j^2(t) r_j^2 . \tag{1}$$

We restrict the discussion to the case of zero temperature, which is a realistic assumption for the experiments in [1,2]. The state of the condensate for a static trap can thus be described using a Hartree-Fock ansatz:

$$|\Psi\rangle = |\Phi\rangle \otimes \cdots \otimes |\Phi\rangle . \tag{2}$$

The minimization of mean energy gives the time independent Gross-Pitaevskii equation for $|\Phi\rangle$:

$$\mu\Phi(\vec{r}) = \left[-\frac{\hbar^2}{2m}\Delta + U(\vec{r},0) + Ng|\Phi(\vec{r})|^2 \right]\Phi(\vec{r}) , \tag{3}$$

with $N - 1 \simeq N$.

In the regime where the atomic interactions are dominant $[Ng|\Phi(\vec{0})|^2 \simeq \mu \gg \hbar\omega_j$ for $j = 1,2,3]$ we can use the Thomas-Fermi approximation to solve (3) [9]; that is, we can neglect the kinetic energy term as compared to the interaction energy term. The result is

$$\Phi(\vec{r}) \simeq \Phi_{TF}(\vec{r}) = \left(\frac{\mu - U(\vec{r},0)}{Ng} \right)^{1/2} , \tag{4}$$

when $\mu \geq U(\vec{r},0)$, and $\Phi(\vec{r}) = 0$ otherwise. The chemical potential μ is determined by the normalization of $|\Phi\rangle$:

$$\mu = \frac{1}{2}\hbar\bar{\omega}\left(15Na\sqrt{\frac{m\bar{\omega}}{\hbar}} \right)^{2/5} , \tag{5}$$

where $\bar{\omega} = [\omega_1(0)\omega_2(0)\omega_3(0)]^{1/3}$.

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One can generalize the Hartree-Fock ansatz (2) to the time dependent case, in which Φ is a function of t. A time dependent variational calculus leads to an (explicitly) time dependent Gross-Pitaevskii equation [10,11]:

$$i\hbar\partial_t\Phi(\vec{r},t) = \Big[-\frac{\hbar^2}{2m}\Delta + U(\vec{r},t) + Ng|\Phi(\vec{r},t)|^2\Big]\Phi(\vec{r},t)\,. \quad (6)$$

In the solution of (6) a Thomas-Fermi type approximation is not directly applicable, because the time variation of the trapping potential would convert potential energy into kinetic energy, which therefore could no longer be neglected. In this paper we identify a unitary transform which eliminates the extra kinetic energy.

We first introduce a model of a classical gas in which each particle experiences a force

$$\vec{F}(\vec{r},t) = -\nabla\big(U(\vec{r},t) + g\rho_{cl}(\vec{r},t)\big)\,, \quad (7)$$

where $\rho_{cl}(\vec{r},t)$ is the spatial density in the gas normalized to N. At $t=0$ the equilibrium condition $\vec{F}=0$ gives $\rho_{cl}(\vec{r},0) = N|\Phi_{TF}(\vec{r},0)|^2$, that is, the classical solution for the steady state density coincides with the quantum solution in the Thomas-Fermi limit. For $t>0$ the exact solution for the classical model can be obtained for the class of potentials (1); in this case, the gas merely experiences a dilatation, any infinitesimally small fraction of the expanding cloud moving along a trajectory

$$R_j(t) = \lambda_j(t)R_j(0) \qquad (j=1,2,3)\,. \quad (8)$$

From this ansatz we obtain for the evolution of the spatial density

$$\rho_{cl}(\vec{r},t) = \frac{1}{\lambda_1(t)\lambda_2(t)\lambda_3(t)} \times \rho_{cl}[\{r_j/\lambda_j(t)\}_{j=1,2,3},0]\,. \quad (9)$$

Newton's law $m\ddot{R}_j(t) = F_j[\vec{R}(t),t]$ applied for the trajectory (8) implies

$$m\ddot{\lambda}_j(t)R_j(0) = -(\partial_{r_j}U)[\vec{R}(t),t] + \frac{1}{\lambda_j\lambda_1\lambda_2\lambda_3}(\partial_{r_j}U)[\vec{R}(0),0] \qquad (j=1,2,3)\,. \quad (10)$$

From Eq. (9) we have expressed the gradient of $g\rho_{cl}(t)$ in terms of $\nabla g\rho_{cl}(t=0) = -\nabla U(t=0)$. For the harmonic potentials U of Eq. (1) both sides of Eq. (10) are proportional to $R_j(0)$. Equation (10) therefore holds for any $\vec{R}(0)$ and the ansatz (8) is self-consistent provided that the scaling factors $\lambda_j(t)$ satisfy

$$\ddot{\lambda}_j = \frac{\omega_j^2(0)}{\lambda_j\lambda_1\lambda_2\lambda_3} - \omega_j^2(t)\lambda_j \qquad (j=1,2,3)\,. \quad (11)$$

The initial conditions are $\lambda_j(0)=1$ and since the gas is initially at rest, $\dot{\lambda}_j(0)=0$. Taking the time derivative of Eq. (8) and eliminating the initial position by Eq. (8) we obtain for the local velocity of the expanding cloud

$$v_j(\vec{r},t) = r_j\frac{\dot{\lambda}_j(t)}{\lambda_j(t)}\,. \quad (12)$$

Equations (11) and (12) do not depend on the interaction strength g. The g dependence is entirely contained in the initial spatial density of the gas [12].

This classical solution motivates the ansatz for the solution of the quantum equation (6):

$$\Phi(\vec{r},t) = e^{-i\beta(t)}e^{im\sum_j r_j^2\dot{\lambda}_j(t)/2\hbar\lambda_j(t)} \times \frac{\tilde{\Phi}[\{r_k/\lambda_k(t)\}_{k=1,2,3},t]}{\sqrt{\lambda_1\lambda_2\lambda_3}}\,. \quad (13)$$

Equation (13) is a unitary transform combining a scaling in $\vec{r}$ and a gauge transform. The gauge transform subtracts from the momentum operator $\hat{\vec{p}}$ the local momentum of the expanding classical gas (12):

$$\hat{p}_j \to \hat{p}_j + m\frac{\dot{\lambda}_j(t)}{\lambda_j(t)}\hat{r}_j\,. \quad (14)$$

The scaling transform mimics the dilatation (9) obtained in the classical case. We insert the ansatz (13) in Eq. (6). For the convenient choice of the global phase factor $e^{-i\beta(t)}$, $\hbar\dot{\beta} = \mu/\lambda_1(t)\lambda_2(t)\lambda_3(t)$, we obtain after some algebra the following time evolution for $\tilde{\Phi}(\vec{r},t)$:

$$\Big[i\hbar\partial_t + \frac{\hbar^2}{2m}\sum_j\frac{1}{\lambda_j^2(t)}\partial_{r_j}^2\Big]\tilde{\Phi}(\vec{r},t) = \frac{1}{\lambda_1(t)\lambda_2(t)\lambda_3(t)}[-\mu + U(\vec{r},0) + Ng|\tilde{\Phi}(\vec{r},t)|^2]\tilde{\Phi}(\vec{r},t)\,, \quad (15)$$

with the initial condition $\tilde{\Phi}(\vec{r},0) = \Phi(\vec{r},0)$. In the Thomas-Fermi regime the right hand side of Eq. (15) is initially very small; the kinetic energy terms on the left hand side are also small initially, and are expected to remain small in time, since the extra kinetic energy due to a change in the trapping potential has been absorbed in the unitary transform (13). We therefore expect that $\tilde{\Phi}(\vec{r},t)$ evolves weakly in time.

To show this point more rigorously we split $\tilde{\Phi}(\vec{r},t)$ into $\Phi(\vec{r},0) + \delta\tilde{\Phi}(\vec{r},t)$. From Eqs. (15) and (3) we find that $\delta\tilde{\Phi}(\vec{r},t)$ obeys a nonlinear inhomogeneous equation with a source term given by

$$S(\vec{r},t) = -\frac{\hbar^2}{2m}\sum_{j=1}^{3}\Big(\frac{1}{\lambda_j^2(t)} - \frac{1}{\lambda_1(t)\lambda_2(t)\lambda_3(t)}\Big)\partial_{r_j}^2\Phi(\vec{r},0)\,. \quad (16)$$

In the Thomas-Fermi approximation the spatial derivatives of $\Phi(\vec{r},0)$ are neglected and the source term vanishes; in this case, $\delta\tilde{\Phi}(\vec{r},t)$ being initially zero remains zero, and $\Phi(\vec{r},t)$ remains constant [13]:

$$\tilde{\Phi}(\vec{r},t) \simeq \Phi(\vec{r},0)\,. \tag{17}$$

This result provides a generalization of the Thomas-Fermi approximation to time dependent problems. All the dynamics of the macroscopic wave function are contained in the evolution of three scaling parameters. In particular the condensate density is a time dependent inverted paraboloid:

$$N|\Phi(\vec{r},t)|^2_{TF} = \frac{\mu - \sum_{j=1}^{3} \frac{1}{2} m\omega_j^2(0) r_j^2/\lambda_j^2(t)}{g\lambda_1(t)\lambda_2(t)\lambda_3(t)}\,, \tag{18}$$

when the right hand side is positive and $|\Phi|^2_{TF} = 0$ otherwise.

We now apply the above results to experimental data obtained in the Ioffe-Pritchard trap of [3]. The trap is axially symmetric with respect to z and cigar shaped ($\omega_1 = \omega_2 \equiv \omega_\perp \gg \omega_3 \equiv \omega_z$).

We consider first the simplest case of a sudden and total opening of the trap at $t = 0$. The equations for the evolution of the scaling parameters (11) simplify to

$$\frac{d^2}{d\tau^2}\lambda_\perp = \frac{1}{\lambda_\perp^3\lambda_z}\,,$$
$$\frac{d^2}{d\tau^2}\lambda_z = \frac{\epsilon^2}{\lambda_\perp^2\lambda_z^2}\,, \tag{19}$$

where $\lambda_\perp$ stands for $\lambda_1 = \lambda_2$ and λ_z stands for λ_3. We have introduced a dimensionless time variable $\tau = \omega_\perp(0)t$ and a parameter $\epsilon = \omega_z(0)/\omega_\perp(0) \ll 1$. We solve (19) by an expansion in powers of ϵ. To zeroth order in ϵ, $\lambda_z = 1$ and the radial expansion scales as

$$\lambda_\perp(\tau) = \sqrt{1 + \tau^2}\,. \tag{20}$$

To second order in ϵ the axial expansion of the cloud is given by

$$\lambda_z(\tau) = 1 + \epsilon^2[\tau \arctan\tau - \ln\sqrt{1+\tau^2}] + O(\epsilon^4)\,. \tag{21}$$

For the experiments considered the term in ϵ^2 is not negligible.

We have performed a fit of the images obtained for two different times of flight in [3]. We used an inverted paraboloid for the density of the cloud, having as free parameters the radial width $W_\perp$, the axial width W_z, and the number of condensed atoms N. Figure 1 shows a cut along the x axis (that is, at $z = 0$) of the spatial density of the cloud integrated along y. According to Eq. (18) the aspect ratio of the cloud is given by

$$\frac{W_z(t)}{W_\perp(t)} = \frac{\lambda_z(t)\sqrt{2\mu/m\omega_z^2(0)}}{\lambda_\perp(t)\sqrt{2\mu/m\omega_\perp^2(0)}} = \frac{\lambda_z(t)}{\lambda_\perp(t)}\,\frac{1}{\epsilon}\,. \tag{22}$$

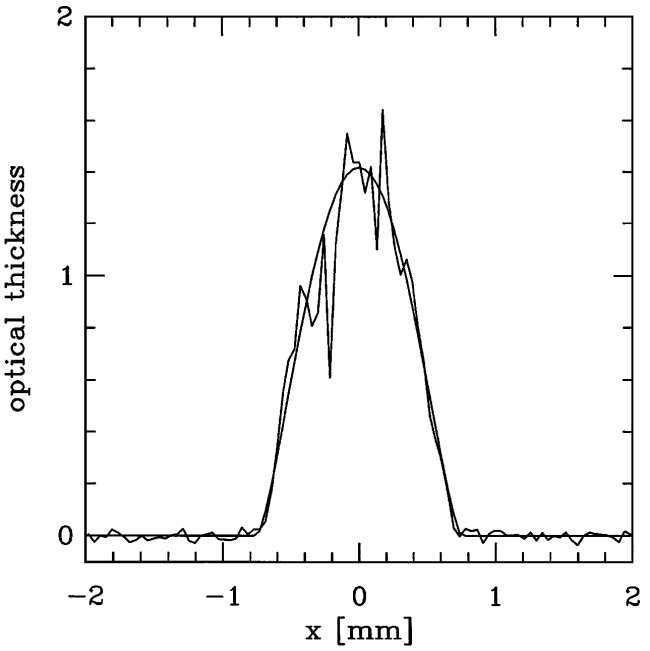

FIG. 1. Spatial density of an expanding condensate integrated along the y axis, cut along the x axis (that is, at $z = 0$). Experimental data obtained at MIT (expansion time of 40 ms) and fit from theory.

The fit gives the values of this ratio for two different expansion times, from which we calculate the two unknown frequencies $\omega_\perp(0)$ and $\omega_z(0)$, using Eqs. (20) and (21). From $W_\perp$ we calculate μ; the relation (5) then leads to a scattering length of $a = 42 \pm 15$ Bohr for sodium, in agreement with earlier measurements [14].

In a second generation of experiments a collective excitation of the condensate has been induced by a time modulation of the eigenfrequencies of the trapping potential [4,5]. In [5] the axial frequency is modulated as $\omega_z^2(\tau) = \omega_z^2(0)[1 - \eta(1 - \cos\Omega\tau)]$ [Ω in units of $\omega_\perp(0)$]. After the excitation the cloud freely oscillates in the unperturbed potential for an adjustable time. Finally, the trapping potential is switched off and the expanding cloud is monitored.

By including this experimental sequence in the evolution of the scaling paameters (11), we give an *ab initio* calculation of the time of flight signals. For a weak modulation ($\eta \ll 1$) the time evolution in the trap is obtained from a linearization of Eq. (11) around the steady state value 1. During the excitation process we obtain for the deviations $\delta\lambda_\perp$ and $\delta\lambda_z$:

$$\frac{d^2}{d\tau^2}\delta\lambda_\perp(\tau) = -4\delta\lambda_\perp(\tau) - \delta\lambda_z(\tau)\,, \tag{23}$$

$$\frac{d^2}{d\tau^2}\delta\lambda_z(\tau) = -2\epsilon^2\delta\lambda_\perp(\tau) - 3\epsilon^2\delta\lambda_z(\tau) + \epsilon^2\eta[1 - \cos(\Omega\tau)]\,. \tag{24}$$

To leading order in ϵ the eigenmodes of this linear system have frequencies [in units of $\omega_\perp(0)$] $\Omega_{\text{fast}} = 2$ and $\Omega_{\text{slow}} = \sqrt{\frac{5}{2}}\epsilon$ and are polarized along $(1, 1, 0)$ and $(1, 1, -4)$, respectively [15,16]. In the experiments the driving frequency Ω was set to Ω_{slow} to achieve a resonant excitation of the slow mode. To lowest order in ϵ, this allows us to keep only the slow mode component of the solution, for which $\delta\lambda_\perp = -\delta\lambda_z/4$. Equation (24) integrated for the time duration τ_e of the excitation then leads to

$$\delta\lambda_z(\tau_e) = \frac{2\eta}{5}[1 - \cos(\Omega_{\text{slow}}\tau_e)] - \frac{\eta}{5}\Omega_{\text{slow}}\tau_e \sin(\Omega_{\text{slow}}\tau_e). \quad (25)$$

In [5] the potential was modulated for five cycles so that $\tau_e = 5(2\pi/\Omega_{\text{slow}})$ and $\delta\lambda_z(\tau_e) = 0$, $\frac{d}{d\tau}\delta\lambda_z(\tau_e) = -2\pi\eta\Omega_{\text{slow}}$. The subsequent evolution in the unperturbed trapping potential is sinusoidal with the eigenfrequency Ω_{slow}; after the free oscillation time τ_o it leads to $\delta\lambda_z(\tau_e + \tau_o) = -4\delta\lambda_\perp(\tau_e + \tau_o) = -2\pi\eta \sin\Omega_{\text{slow}}\tau_o$.

Finally the trapping potential is switched off to monitor the excited condensate. The time evolution of the scaling parameters is obtained by linearizing Eq. (19) around the solutions Eqs. (20) and (21) with initial conditions given by the values of $\delta\lambda_{z,\perp}$ and their derivatives at $\tau = \tau_e + \tau_o$. After a time of flight τ_f we obtain, neglecting terms of order 2 in ϵ and terms vanishing in the limit $\tau_f \to \infty$,

$$\delta\lambda_\perp(\tau_e + \tau_o + \tau_f) = -\frac{1}{4}\tau_f\delta\lambda_z(\tau_e + \tau_o) - \frac{1}{4}[\pi\tau_f - 4\ln\tau_f + 1]\frac{d}{d\tau}\delta\lambda_z(\tau_e + \tau_o), \quad (26)$$

$$\delta\lambda_z(\tau_e + \tau_o + \tau_f) = \delta\lambda_z(\tau_e + \tau_o) + \tau_f\frac{d}{d\tau}\delta\lambda_z(\tau_e + \tau_o). \quad (27)$$

From this we determine the aspect ratio (22) of the expanding cloud. Figure 2 shows that our predictions are in good agreement with the experimental results of [5] for short free oscillation times τ_o. For longer times τ_o a damping of the oscillations is observed experimentally, which cannot be explained with our mean field treatment.

In conclusion, we have been able to extend the Thomas-Fermi approximation to the motion of a condensate in a time dependent harmonic potential: the time dependence is entirely contained in three scaling factors which can be obtained from the evolution of a classical gas. This provides an easy quantitative tool for the analysis of current experiments on trapped condensed gases. We have applied it to two recent experiments. From time of flight images of the condensate at two different expansion times we could calculate the scattering length a without relying on measurements of the trapping frequencies. For collective excitations of a condensate in the linear response regime we could predict not only the frequency but also the phase and amplitude of the measured signal (see Fig. 2). Our treatment can be applied in the nonlinear response regime as well, for example, for oscillations of the condensate induced by a strong modulation or by a partial opening of the trap.

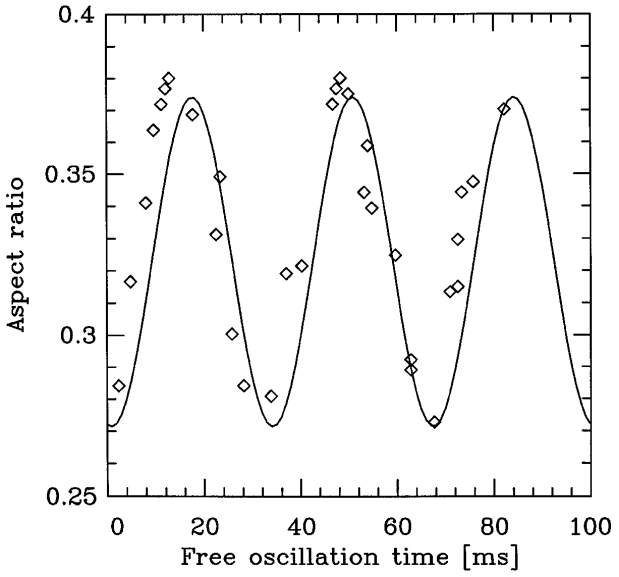

FIG. 2. Aspect ratio of the excited and expanded condensate as a function of free oscillation time τ_o. Expansion time $\tau_f = 40$ ms, $\omega_\perp(0) = 2\pi \times 250$ Hz, $\omega_z(0) = 2\pi \times 19$ Hz, $\eta = 0.005$. Solid line: theory. Diamonds: experimental data obtained at MIT.

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