Higher Dimensional Realizations of Activated Dynamic Scaling at Random Quantum Transitions

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We show that many of the unusual properties of the one-dimensional random quantum Ising model are shared also by dilute quantum Ising systems in the vicinity of a certain quantum transition in *any* dimension $d > 1$. Thus while these properties are not an artifact of $d = 1$, they do require special circumstances in higher dimensions. [S0031-9007(96)01936-9]

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There is considerable current interest in the properties of phase transitions in random quantum systems. Experimentally accessible quantum transitions such as the transition from an insulator to a metal [1] or a superconductor [2], and various magnetic-nonmagnetic transitions in heavy fermion compounds [3], high-T_c cuprates [4], or insulating dipolar Ising magnets [5], often occur in situations with strong randomness, and are only poorly understood. Theoretically many authors [6–8] have analyzed the random Ising chain in a transverse field—perhaps the simplest random quantum system. In particular, Fisher [8] used a real space renormalization group (RG) approach to obtain a detailed description of the thermodynamics and static correlation functions in the vicinity of the critical point. The properties of the system were found to be very unusual as compared to conventional quantum critical points. Specifically, length scales were found to diverge logarithmically with energy scales, implying a dynamic critical exponent $z = \infty$. The transition was shown to be flanked on either side by "Griffiths" regions with a susceptibility diverging due to contributions from statistically rare fluctuations. There are very few reliable results on other random quantum transitions, especially in finite dimensions $d > 1$; thus it is important to understand if the anomalous properties of the random quantum Ising chain are a specialty of $d = 1$, or if there exist higher dimensional quantum transitions which share these properties. Numerical work on higher dimensional transverse field Ising spin glasses has found evidence for the presence of Griffiths regions, but the dynamic scaling at the critical point seems conventional [9].

In this paper, we provide a simple example of such anomalous scaling in higher dimensional random quantum Ising systems. We consider bond or site diluted Ising models with short range interactions. As was first suggested by Harris [10], and as we argue below, these models have two quantum transitions (see Fig. 1): At low dilution, below the percolation threshold, there is a phase transition when the long range ferromagnetic order is destroyed by increasing the transverse field. This is expected to be in the universality class of the generic random bond quantum Ising transition. Right at the percolation threshold, there is a finite range of transverse field strengths at which the system remains critical. There is thus another quantum transition, across the percolation threshold, at low but nonzero transverse field strengths, which is potentially in a different universality class. These two critical lines meet at a multicritical point (M). The properties of the second transition, at the percolation threshold, are determined largely by the statistics and geometry of the percolating clusters about which much information is available. This permits us to make definitive statements about the scaling properties of this transition in any dimension. We show that the dynamic scaling is activated, with length scales diverging logarithmically with energy scales. We also demonstrate the existence of Griffiths phases on either side with diverging susceptibility. Our approach shows clearly the connection between these behaviors, and $T = 0$ phase transitions at which quantum fluctuations are "dangerously irrelevant" [8,11]; indeed, various exponents of the transition are given by those of

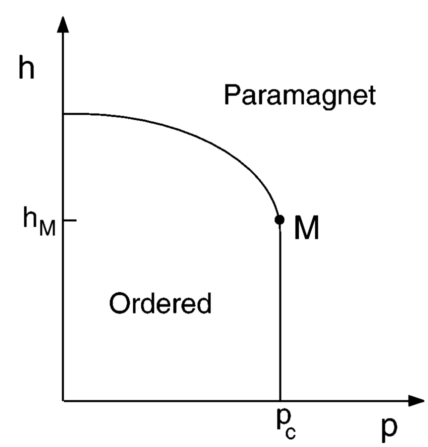

FIG. 1. Phase diagram of the dilute Ising model in a transverse field (h) at zero temperature. The dilution probability is p. The multicritical point M is at $p = p_c$, $h = h_M$. The quantum transition along the vertical phase boundary ($h < h_M$, $p = p_c$) is controlled by the classical percolation fixed point at $p = p_c$, $h = 0$; quantum effects (due to a nonzero h) are dangerously irrelevant and lead to activated dynamic scaling near the $h < h_M$, $p = p_c$ line.

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the classical percolation transition and are hence known either exactly or numerically. We also obtain bounds on exponents characterizing the multicritical point.

For concreteness, we consider *bond*-diluted Ising models defined by the Hamiltonian

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}\sigma_i^z\sigma_j^z - \sum_i h\sigma_i^x, \qquad (1)$$

where $\langle ij\rangle$ labels the nearest-neighbor sites of a d-dimensional lattice ($d > 1$), and J_{ij} equals 0 with probability p and equals $J > 0$ with probability $1 - p$. The transverse field h is nonrandom, although our results remain valid for weakly random distributions as well. At $p = 0$, as h is increased, there is a zero temperature phase transition from an ordered ground state to a disordered one (see Fig. 1). On the other axis, when $h = 0$, so that the system is classical, there is a percolation transition at $p = p_c$. For $p < p_c$ there is a thermodynamically large connected cluster, while for $p > p_c$ there are only finite connected clusters. For $p < p_c$, for small enough h, the system retains long range order. This is ultimately destroyed for some $h > h_c(p)$, with $h_c(p)$ expected to be a monotonically decreasing function of p. On the other hand if $p > p_c$, there is no long range order for any h.

Now consider $p = p_c$. Again there is zero magnetization and no long range order for any h. However, the system stays critical for $h < h_M = h_c(p_c)$ (Fig. 1). To see this, note that although there is no thermodynamically large connected cluster at p_c, there still is an infinite connected cluster with a fractal dimension smaller than d. The spins on this cluster align together at $h = 0$. A small but nonzero h is not sufficient to destroy this order on the critical cluster. In fact, two spins on any sufficiently large finite cluster remain strongly correlated with each other for small h. The critical cluster eventually loses order when h reaches h_M. To make this more precise consider the disorder averaged, equal time (τ), two point spin correlation function. Spins at points 0 and x are correlated only if they belong to the same cluster; thus $G(x) \equiv [\langle\sigma^z(x,\tau = 0)\sigma^z(0,0)\rangle - \langle\sigma^z(x,0)\rangle\langle\sigma^z(0,0)\rangle] = \int dC C\mathcal{P}$ (0 and x belong to the same cluster) $\mathcal{P}$ (if 0 and x belong to the same cluster, they have correlation C), where angular brackets are averages over quantum/thermal fluctuations, square brackets represent disorder averages, and $\mathcal{P}(E)$ is the probability of the event E. At $p = p_c$, the first probability $\sim x^{-d+2-\eta_p}$ for large x [12]. The arguments above imply that the integral over C is nonzero and independent of x for large x for $h < h_M$. Thus $G(x, p = p_c) \sim x^{-d+2-\eta_p}$ for large x for $h < h_M$, and there is a critical line for $h < h_M$ at $p = p_c$.

In this paper we will primarily focus attention on the transition across this critical line. We show that at $T = 0$ the properties of this transition are strikingly similar to the $d = 1$ results for the random quantum Ising chain [8]. First consider the equal-time two point function away from p_c. Arguing as in the previous paragraph, we get $G(x) \sim \mathcal{P}$ (0 and x belong to the same cluster) $\sim x^{-d+2-\eta_p} f(x/\xi)$ for large x where the correlation length $\xi \sim |p - p_c|^{-\nu_p}$. Similarly, for $p < p_c$ the mean magnetization $[\langle\sigma^z(x,0)\rangle] \sim \mathcal{P}$ (any given site belongs to infinite cluster) $\simeq (p_c - p)^{\beta_p}$. The exponents β_p, η_p, ν_p are those of classical percolation theory, and are known exactly in $d = 2$ and for $d > 6$. For any $d < 6$, they satisfy $\beta_p = \nu_p(d - 2 + \eta_p)/2$.

Now consider dynamic correlations. The imaginary part of the local dynamic susceptibility $[\chi_L''(\omega)] = \Sigma_N \mathcal{P}$ (given site i belongs to a cluster of N sites)$\int y dy \mathcal{P}$ [if site i belongs to a cluster of N sites, $\chi_i''(\omega) = y$]. The energy levels of a cluster of N sites can be described for $h \ll J$ as follows: The two lowest levels correspond to the states of a single effective Ising spin with magnetic moment $\sim N$ in an effective transverse field $h_{\mathrm{eff},N}$. For large N, $h_{\mathrm{eff},N}$ can be estimated in Nth order perturbation theory to be $\tilde{h}\exp(-cN)$. (In general, there would also be a prefactor that varies as a power of N; we will ignore this as it is subdominant to the exponential in N. When necessary, we will indicate the modifications induced by keeping this prefactor.) The quantities $\tilde{h}$ and c are of order h and $\ln(J/h)$, respectively, but their precise values depend on the particular cluster being considered. As the distribution of $\tilde{h}$ and c is not expected to become very broad near the transition [13], we will replace them by their typical values h_0 and c_0, respectively. Apart from these two lowest levels, there are other levels separated from these by energies $\sim J$. These can be ignored for the low-energy physics, and for small $\omega \ll h$, we only need to consider large clusters.

We now need the following results [12] from percolation theory for the probability $P(N,p)$ that a given site belongs to a large cluster of N sites. At $p = p_c$, $P(N, p_c) \sim N^{1-\tau}$ where τ equals $\frac{5}{2}$ for $d > 6$, and equals $1 + d/D$ for $d < 6$ where D is the fractal dimension of the critical clusters. In particular $\tau = 187/91$ in $d = 2$, and is approximately 2.18 in $d = 3$. Away from p_c, for $d < 6$ or $d > 8$, $P(N,p)$ satisfies the scaling form

$$P(N,p) \sim N^{1-\tau} g(N/\xi^D). \qquad (2)$$

The scaling function $g(y)$ is universal but is different for $p < p_c$ and $p > p_c$; it approaches 1 for $y \ll 1$, while for $y \gg 1$,

$$\begin{aligned} g(y, p > p_c) &\sim y^{-\theta+\tau} e^{-c_+ y}, \\ g(y, p < p_c) &\sim y^{-\theta'+\tau} e^{-c_- y^{1-1/d}}, \end{aligned} \qquad (3)$$

with $\theta = 1$, $\frac{3}{2}$ and $\frac{5}{2}$ for $d = 2, 3$ and $d > 8$, respectively, and $\theta' = \frac{5}{4}$, $-1/9$ in $d = 2, 3$, respectively. The constants $c_{+,-}$ are of order unity. For $6 < d < 8$, $P(N,p)$ satisfies a more complicated two-variable scaling form [14]. For simplicity, we shall not discuss this case here, though including it is straightforward.

Using these percolation results, we get for the dynamic susceptibility of the Ising model

$$[\chi_L''(\omega)] \sim \int \frac{dN}{N^{\tau-1}} g(N/\xi^D)\delta(\omega - h_0 e^{-c_0 N}) \sim \frac{1}{\omega[\ln(h_0/\omega)]^{\tau-1}} g\left[\frac{\ln(h_0/\omega)}{c_0\xi^D}\right]. \quad (4)$$

Note that the scaling variable is $\ln(1/\omega)/\xi^D$. This is a precise statement of the activated dynamic scaling mentioned earlier, which has thus been shown to occur in all $d > 1$ in the present model. Asymptotic forms in various limits can be obtained from the limiting behavior of $g(x)$ described above. For $p \geq p_c$, we get $\chi_L'' \sim \omega^{-1+\alpha}[\ln(h_0/\omega)]^{1-\tau}$ with $\alpha \sim \xi^{-D}$ so that at $p = p_c, \alpha = 0$ (including the power-law prefactor in $h_{\mathrm{eff},N}$ will only change the power of $\ln(1/\omega)$ in by $\sim\xi^{-D}$, and similarly for the prefactor in the expression for $p < p_c$ given below). Note that just on the disordered side the system is gapless with a power-law density of states. The physical origin of this is, as usual, the presence of rare, large clusters with arbitrarily small energy gaps. For $p < p_c$, the presence of the infinite cluster (and the associated long range order) gives rise to a delta function at $\omega = 0$. For $\omega \neq 0$, $\chi_L''(\omega)$ is still determined by contributions from the finite clusters. Proceeding as before, we find $\chi_L''(\omega \neq 0) \sim (1/\omega)[\ln(h_0/\omega)]^{1-\tau}\exp\{-\kappa[\ln(h_0/\omega)]^{1-1/d}\}$ with $\kappa \sim \xi^{-D(1-1/d)}$. Again the system is gapless. The gaplessness of both the ordered and disordered phases in the vicinity of the transition is unlike quantum transitions in pure systems, but is probably generic to many random quantum transitions [8,9,15].

The magnetization in response to a uniform external applied magnetic field H along the $\hat{z}$ direction can be calculated similarly. For small $H \ll h$, only large clusters contribute. The magnetization per site of a cluster of size N is that of an Ising spin of magnetic moment N in a transverse field $h_{\mathrm{eff},N}$, and is therefore given by

$$M_N(H) = \frac{NH}{[(NH)^2 + h_{\mathrm{eff},N}^2]^{1/2}}.$$

Thus the total magnetization per site (after subtracting the regular contribution of the infinite cluster for $p < p_c$) is

$$M(H) - M(H = 0) \sim \int dN \frac{1}{N^{\tau-1}} g(N/\xi^D) M_N(H).$$

The singular part therefore has the scaling form

$$M_{\mathrm{sing}}(H) \sim \frac{1}{[\ln(h_0/H)]^{\tau-2}} \Phi\left[c\frac{\ln(h_0/H)}{\xi^D}\right], \quad (5)$$

with c a nonuniversal constant, and $\Phi(y)$ a universal function which is related to $g(y)$ by

$$\Phi(y) = \int_1^\infty w^{1-\tau}\, dw g(wy).$$

Again note that the scaling variable is $\ln(h_0/H)\xi^{-D}$, which is unlike conventional critical behavior, but is similar to the $d = 1$ random quantum Ising chain [8].

We thus have the following asymptotic forms as $H \rightarrow 0$:

$$M_{\mathrm{sing}}(H) \sim \begin{cases} [\ln(h_0/H)]^{2-\tau}, & p = p_c, \\ \xi^D[\ln(h_0/H)]^{1-\theta}(H/h_0)^\gamma, & p \geq p_c, \\ \xi^{D(1-1/d)}[\ln(h_0/H)]^{-\theta'+1-1/d} e^{-[\gamma' \ln(h_0/H)]^{1-1/d}}, & p \leq p_c, \end{cases} \quad (6)$$

with $\gamma, \gamma' \sim \xi^{-D}$ [as with the dynamic susceptibility, including the power-law prefactor in $h_{\mathrm{eff},N}$ changes the power of $\ln(1/H)$ for $p > p_c$, and the prefactor for $p < p_c$ by $\sim\xi^{-D}$]. Note that the magnetization rises as a power of H, with a continuously varying exponent which is smaller than one in a region of the disordered phase close to the transition. This is again similar to the $d = 1$ result and gives rise to a divergent linear susceptibility throughout this region. In the ordered side $dM/dH \sim 1/H$ with weak corrections. Thus the linear susceptibility diverges in the ordered side as well.

Similarities to $d = 1$ are also present for the random q-state Potts model. In $d = 1$, all the critical exponents and the scaling functions for the mean spatial correlations and magnetization are independent of q [16], i.e., they are the same as for the Ising case. For the $d > 1$ diluted models, it should be clear that a vertical critical line exists at the percolation threshold for all q. The exponents and appropriate scaling functions of the corresponding quantum transitions are again independent of q as they are determined mainly by the geometric properties of the lattice near the percolation threshold. As in $d = 1$, all the q dependence is in nonuniversal quantities and in a high-energy cutoff limiting the regime of universal scaling behavior.

We now briefly mention results for $T \neq 0$. At $p = p_c$ and $h < h_M$ there is a thermal correlation length $\xi_T \sim \exp(\mathrm{const}/T)$, much like the $h = 0$ classical model [10], and using similar arguments. In contrast, for the $d = 1$ random quantum Ising chain, ξ_T rises as a power of $\ln(1/T)$ at the critical coupling. For $p > p_c$ and T below the crossover point $\xi \sim \xi_T$, or $T < \ln^{-1}[1/(|p - p_c|]$, we found, for instance, a linear susceptibility $\chi_T \sim T^{-1+\kappa}$ (up to log corrections) with $\kappa \sim \xi^{-D}$. For $p < p_c$ there is a phase transition at $T_c \sim \ln^{-1}[1/(p_c - p)]$, as in the classical model.

We now turn to the special point M (Fig. 1). Correlations at the point M should decay faster than along the vertical critical line considered above. Thus if $G_M(x) \sim x^{-\phi}$, then $\phi > d - 2 + \eta_p$. Similarly, as p approaches

p_c from below at h_M, the magnetization should go to zero faster than for $h < h_M$. Thus $\beta_M > \beta_p$.

Finally, consider diluted quantum $O(N)$ rotor models. For $N > 2$, the presence of one-dimensional segments in the critical clusters implies rapidly decaying correlations in large clusters. Thus the vertical critical line will not be present for $N > 2$. The $O(2)$ case is special as correlations decay only as a power-law in one dimension. Whether this is like the Ising or the $O(N > 2)$ system we leave as an open question.

To summarize, we have presented a simple example of a random quantum transition in $d > 1$ which exhibits many of the properties of the transition in the $d = 1$ random quantum Ising chain. In particular, the dynamic scaling was activated, with $\ln(1/\text{energy scale}) \sim \xi^D$. Further, there were Griffiths regions on either side of the transition, with a singular density of states and a diverging susceptibility. It would be interesting to find experimental systems where this transition can be studied. (One possibility is the system $\mathrm{LiHo}_x\mathrm{Y}_{1-x}\mathrm{F}_4$ where, however, the presence of dipolar interactions may complicate the situation.) Theoretically an important feature of this transition, as in the $d = 1$ system, is that it is controlled by a classical fixed point with quantum fluctuations being dangerously irrelevant. This feature is also found in quantum Ising models in a random longitudinal field, which undergo, for $d > 2$, a phase transition from an ordered to a paramagnetic phase at $T = 0$ [17]. In fact, this fixed point also controls the nonzero T phase transition where it has been argued that the classical dynamic scaling is activated [18]. Extending the argument to the quantum dynamics at the $T = 0$ transition suggests that the quantum dynamic scaling would again be activated [19]. Further, classical fixed points have also appeared in recent studies of the metal-insulator transition [20], and in the $T = 0$ onset of spin-glass order in metallic systems [11]; whether in these cases the dynamic scaling is activated or conventional due to the presence of itinerant fermions is worthy of future investigation.

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