Random Matrix Model for Superconductors in a Magnetic Field

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We introduce a random matrix ensemble for bulk type-II superconductors in the mixed state and determine the single-particle excitation spectrum using random matrix theory. The results are compared with planar tunnel junction experiments in PbBi thin films. More low energy states appear than in the Abrikosov-Gor'kov-Maki or Ginzburg-Landau descriptions, consistent with observations. [S0031-9007(96)01906-0]

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Random matrices have been used to understand the distribution of level spacings and widths in nuclei [1,2], complex atoms [2], small metallic particles [3,4], and quantum systems whose classical limits are chaotic [5]. Random matrix models have also appeared in the context of solving certain SU(N)-invariant field theories in the large N limit [6,7] and discretizing two-dimensional quantum gravity [8]. In this paper we consider a different context: using a random matrix model to solve the electronic structure problem posed by the BCS description of a superconductor in a magnetic field.

The BCS description for an ideal electronic system, and the more general description for a realistic system, can be formulated in a particular basis in terms of a large matrix in which the matrix elements have rapidly varying phases and a smooth magnitude distribution. This problem, however, is too complicated to solve exactly. For an ideal system, the matrix is too large, and for a realistic system, the matrix elements are unknown. There are two motivations for using a random matrix model: (1) In the limit of small level spacings compared to energy scales of interest, spectra are often insensitive to the details of matrix elements with uncorrelated phases and the same average magnitude. The robustness of the Wigner semicircle distribution is an example. (2) A simple set of integral equations may be derived for the spectrum. These equations may be solved numerically and compared directly with experiments.

The microscopic model for a superconductor in a magnetic field was developed as a generalization of the zero-field BCS [9] theory by Gor'kov [10], using a Green's function description, and by de Gennes [11], using a wave-function description. The variational Hamiltonian is

$$\mathcal{H}' = \int d\mathbf{r}\, \Psi_{\mathbf{r}}^{\dagger} \begin{bmatrix} \mathcal{H}_0(\mathbf{r}) & \Phi(\mathbf{r}) \\ \Phi^*(\mathbf{r}) & -\mathcal{H}_0^*(\mathbf{r}) \end{bmatrix} \Psi_{\mathbf{r}}\,, \qquad (1)$$

where $\Psi_{\mathbf{r}}^{\dagger} \equiv [c_{\mathbf{r}\uparrow}^{\dagger} c_{\mathbf{r}\downarrow}]$, $c_{\mathbf{r}\sigma}^{\dagger}$ are the electron creation operators, and $\mathcal{H}_0$ is the bare Hamiltonian $\mathcal{H}_0 \equiv (1/2m)(i\nabla - e\mathbf{A}/c)^2 - E_F$. The order parameter is

$$\Phi(\mathbf{r}) = -V_0 \Omega \langle c_{\mathbf{r}\uparrow} c_{\mathbf{r}\downarrow} \rangle \qquad (2)$$

for a system in a volume Ω and with a local two-body interaction of strength $V_0 > 0$. In the absence of a magnetic field, plane waves diagonalize the bare Hamiltonian and the order parameter may be taken to be a constant, $\Phi(\mathbf{r}) = \Delta_0$. Equation (1) then separates into 2×2 matrices, yielding the BCS spectrum $E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_0^2}$, where $\varepsilon_{\mathbf{k}} = \hbar^2 k^2/2m - E_F$.

In magnetic fields large compared to the field of first penetration, H_{c1}, the field inside the superconductor is nearly uniform. The eigenstates of $\mathcal{H}_0$, for an ideal system, are Landau levels. The momentum with respect to the vortex lattice, which distinguishes states within a Landau level, is conserved, and Eq. (1) separates into $2N \times 2N$ matrices where $N \sim \omega_D/\omega_c$, the cutoff energy for the pairing interaction over the Landau level spacing. This formulation has been used to calculate certain electronic properties of an ideal system [12,13].

A more general method [14], which does not depend on the exact eigenstates being Landau levels, is to consider, as in the Anderson description of dirty superconductors in no magnetic field [15], arbitrary eigenstates ψ_α of the bare Hamiltonian:

$$\mathcal{H}_0 \psi_\alpha(\mathbf{r}) = \varepsilon_\alpha \psi_\alpha(\mathbf{r})\,. \qquad (3)$$

New quasiparticle operators may be defined with respect to this basis: $d_{\alpha\sigma}^{\dagger} = \int d\mathbf{r}\, \psi_\alpha(\mathbf{r}) c_{\mathbf{r}\sigma}^{\dagger}$. The order parameter may be written

$$\Phi(\mathbf{r}) \equiv \phi\, \chi(\mathbf{r})\,, \qquad (4)$$

where $\chi(\mathbf{r})$ is normalized so that $\int |\chi(\mathbf{r})|^2 d\mathbf{r} = \Omega$ and ϕ, the spatial average, is real and positive. We can then define a pairing matrix

$$\mathcal{A}_{\alpha\beta} \equiv \int d\mathbf{r}\, \chi(\mathbf{r}) \psi_\alpha(\mathbf{r}) \psi_\beta(\mathbf{r})\,, \qquad (5)$$

so that the variational two-body Hamiltonian is

$$\mathcal{H}' = \Psi^{\dagger} \begin{bmatrix} \varepsilon & \phi\mathcal{A} \\ \phi\mathcal{A}^{\dagger} & -\varepsilon \end{bmatrix} \Psi\,, \qquad (6)$$

where Ψ is the vector of quasiparticle operators $\Psi^{\dagger} \equiv [\cdots d_{\alpha\uparrow}^{\dagger} \cdots \cdots d_{\beta\downarrow} \cdots]$ and ε is the diagonal matrix of bare eigenvalues $|\varepsilon_\alpha| < \hbar\omega_D$. The cutoff energy for the

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pairing interaction $\hbar\omega_D$ is taken to be the Debye energy in conventional superconductors.

In zero magnetic field, time-reversal invariance in an isotropic superconductor ($\chi = 1$) ensures that the pairing matrix connects only time-reversed states: $\mathcal{A}_{\alpha\beta} \propto \delta_{\alpha,\beta}$. In the case of the ideal system, near H_{c2}, it has been shown that $\mathcal{A}_{\alpha\beta} \propto \exp(-|\varepsilon_\alpha - \varepsilon_\beta|^2/W_0^2)$ where $W_0 \approx \Delta_0$, the zero-field gap [12,13]. In other words, the effect of the magnetic field is to "fuzz" out the energy range over which states are paired, by an amount of order of the zero-field gap.

The above pairing matrix description [12–14] motivates the choice of the following model. We consider an ensemble of $2N \times 2N$ Hermitian matrices

$$H = \begin{bmatrix} E_0 & \phi A \\ \phi A^\dagger & -E_0 \end{bmatrix}, \tag{7}$$

where E_0 is a diagonal matrix of N uniformly distributed eigenvalues $|\varepsilon_i| < 1$, and the pairing matrix A is

$$A_{ij} \equiv \frac{1}{\sqrt{N}} h(\varepsilon_i - \varepsilon_j) c_{ij}\,, \tag{8}$$

where c_{ij} are selected from a complex Gaussian random distribution $\langle c_{ij}^* c_{kl}\rangle = \delta_{ik}\delta_{jl}$. The cutoff function h is

$$h(x) \equiv \frac{1}{\pi}\frac{W}{x^2 + W^2}\,. \tag{9}$$

The matrix E_0 models the spectrum of the normal metal in the range $E_F - \hbar\omega_D < E < E_F + \hbar\omega_D$; all energies have been normalized in units of the Debye energy. The bare energy level spacing $\delta = \hbar\omega_D/N$ determines the size of the matrix. The level spacing is assumed to be small, so that the $N \to \infty$ limit applies. The cutoff function h is related to the Fourier transform of the time dependence of the pair correlation function. The Lorentzian form used in Eq. (9) is expected on general grounds in a diffusive system (Ref. [11], Chap. 8). The spectrum is, however, not very sensitive to this choice of cutoff function: a Gaussian cutoff yields results which differ by $\lesssim 10\%$.

The two parameters of the model are ϕ and W. ϕ is the spatial average of the superconducting order parameter. It decreases from the gap value in zero field, $\phi(H = 0) = \Delta_0$, to zero at the phase transition, $\phi(H_{c2}) = 0$. W is the scale for the relative energy difference between electrons being paired. Time-reversal symmetry gives $W(H = 0) = 0$; the scale increases steadily with magnetic field, to $W(H_{c2}) \equiv W_0$. Solving the full self-consistent equations for a given material would, in principle, determine the best values for ϕ and W as a function of applied field and properties of that material. This would also make the model much more complex. We therefore, instead, find solutions for arbitrary ϕ and W and consider these as free parameters to be determined from the data.

The spectrum of H may be determined using methods similar to those discussed in the context of solving large N (1 + 1)-dimensional QCD [6] or (0 + 1)-dimensional matrix-valued φ^4 theory [7]. These techniques were further developed in the context of a large N Anderson disorder model (many orbitals per lattice site) by Wegner [16] and in analyzing the eigenvalue correlators of various matrix models by Brezin and Zee [17].

In general, the connection between random matrix models and (0 + 1)-dimensional field theory follows from a Lagrangian of the form $L = L_0 + L'$, where the bare Lagrangian contains an N vector of fermions ψ and an $N \times N$ scalar matrix M, $L_0 = i\psi^\dagger\dot{\psi} + \mathrm{Tr}M^2$, and the interaction is $L' = N^{-1/2}\psi^\dagger M\psi$. In the large N limit, the fermion propagator for this theory is the ensemble-averaged Green's function for M: $\int dt\, e^{iEt}\langle\psi_i^\dagger(t)\psi_j(0)\rangle = \langle(E - M)_{ij}^{-1}\rangle$. This propagator may be evaluated using standard Feynman diagram techniques. The important result is that in the large N limit only the planar (noncrossing; generalized rainbow) diagrams survive [6,7,16,17].

Some additional methods are necessary for the ensemble defined by Eqs. (7)–(9) because of the Pauli matrix structure and the cutoff function in A. The Green's function for H, $\mathbf{G}(E) \equiv 1/(E - H)$, may be written in the Gor'kov notation,

$$\mathbf{G} \equiv \begin{bmatrix} G & F \\ F' & G' \end{bmatrix} \tag{10}$$

Bold-faced quantities are 2×2 matrices in electron-hole space. Dyson's equation is

$$\mathbf{G} = \mathbf{G}_0 + \mathbf{G}_0\Sigma\mathbf{G}\,, \tag{11}$$

where Σ is the self-energy and $\mathbf{G}_0$ is the bare Green's function ($\phi = 0$). In the large N limit, only the noncrossing diagrams survive, which implies

$$\begin{aligned}\Sigma_{ij} &= \phi^2 \sum_{k,l}\langle A_{ik}A_{jl}^*\rangle \tau_1 \mathbf{G}_{kl}\tau_1 \\ &= \delta_{ij}\frac{1}{N}\phi^2\sum_k \tau_1\mathbf{G}_{kk}\tau_1 h(\varepsilon_k - \varepsilon_i) \\ &\equiv \delta_{ij}\phi^2\tau_1\mathbf{S}(\varepsilon_i)\tau_1\,,\end{aligned}$$

where τ_1 is the Pauli matrix and we introduce the function

$$\mathbf{S}(\varepsilon) \equiv \frac{1}{N}\sum_k \mathbf{G}_{kk} h(\varepsilon_k - \varepsilon)\,. \tag{12}$$

Note that $\mathbf{S}$ depends on two energies, E and ε. It is not a Green's function, it is just a construct useful for solving this particular problem. In the nonbanded, uniform A case we would have $h = 1$ and $\mathbf{S} = (1/N)\mathrm{Tr}\,\mathbf{G}$, independent of ε.

Inserting the result for Σ into Dyson's equation, performing the sum in Eq. (12), and replacing the discrete bare energies ε_i by the continuous variable ε, yields

$$\mathbf{S}(\nu) = \int_{-1}^{1} d\varepsilon\, \frac{h(\nu - \varepsilon)}{[\mathbf{G}_0^{-1}(\varepsilon) - \phi^2\tau_1\mathbf{S}(\varepsilon)\tau_1]}\,. \tag{13}$$

This is an implicit integral equation for $\mathbf{S}$. Writing

$$\mathbf{S} \equiv \begin{bmatrix} S_1 & S_3 \\ S_4 & S_2 \end{bmatrix} \tag{14}$$

and using $\mathbf{G}_0^{-1} = E - \tau_3\varepsilon$, we find that the off-diagonal elements of the matrix in the denominator of the integral equation are S_3 and S_4. Hence, a solution exists with $S_3 = S_4 = 0$. For a given theory, the spectrum in the large N limit is unique, so we expect this to be the only solution. Equation (13) then reduces to

$$S_1(\nu) = \int_{-1}^{1} d\varepsilon \, \frac{h(\nu - \varepsilon)}{E - \varepsilon - \phi^2 S_2(\varepsilon)}, \tag{15a}$$

$$S_2(\nu) = \int_{-1}^{1} d\varepsilon \, \frac{h(\nu - \varepsilon)}{E + \varepsilon - \phi^2 S_1(\varepsilon)}. \tag{15b}$$

For a given ϕ and W, Eq. (15) may be solved numerically to obtain $S_1(\varepsilon)$ and $S_2(\varepsilon)$ at any energy E. The single-particle excitation spectrum then follows from

$$\mathrm{Tr}\, G(E) = \int_{-\infty}^{\infty} d\nu \, S_1(\nu, E), \tag{16}$$

and the usual relation $\rho(E) = (-1/\pi)\,\mathrm{Im}\,\mathrm{Tr}\, G(E + i\varepsilon)$.

In the weak coupling limit, the bare BCS gap $\Delta_0 \ll \hbar\omega_D$, so both ϕ and W in Eq. (15) are much less than 1. In that case, the limits of integration may be extended to ∞ in both directions, W may be scaled out, and solutions depend only on the parameter ϕ/W.

In Fig. 1 we compare the spectrum obtained this way with tunneling spectra from planar tunnel junctions in PbBi/Ge thin films [18]. These spectra where taken at temperatures low enough ($T = 360$ mK) that thermal smearing affects the line shape by less than a few percent. This thermal correction is included with the standard expression [19] for the tunneling current $dI/dV = -(G_0/\rho_0)\int dE\, \rho(E)\partial f(E + eV)/\partial V$, where the Fermi factor is $f(x) = 1/(e^{\beta x} + 1)$.

The numerical agreement between the random matrix model and the tunneling data is fairly close, a little worse at lower fields. Also shown in Fig. 1 is a fit with an Abrikosov-Gor'kov spectrum to the 3 T data, in which the gap and pair-breaking strength are taken as free parameters. A pair-breaking strength [20] $\zeta = 0.5$ was used, which correctly reproduces the peak; a pair-breaking strength $\zeta \approx 1$ gives more low energy states but fits poorly at all energies.

One interesting feature of the random matrix model, as seen in Fig. 1, is the presence of more low energy states at low fields than is conventionally assumed. The conventional view, however, has been based on several indirect arguments rather than a direct solution of a microscopic model.

First, the Abrikosov-Gor'kov solution for the Green's functions of a superconductor in the presence of a dilute gas of magnetic impurities [21] was applied to type–II superconductors by Maki, with impurity concentration mapping to magnetic field strength [20]. This mapping assumes, in addition to the dirty limit and local electrodynamics, translationally invariant Green's functions (e.g., spatially uniform order parameter), which cannot apply to a bulk type-II superconductor at any magnetic field above H_{c1}. Further, it requires that the impurity averaging technique, valid for a dilute gas of weak, uncorrelated, scatterers apply to the case of a nearly uniformly penetrating magnetic field distribution, which seems unlikely.

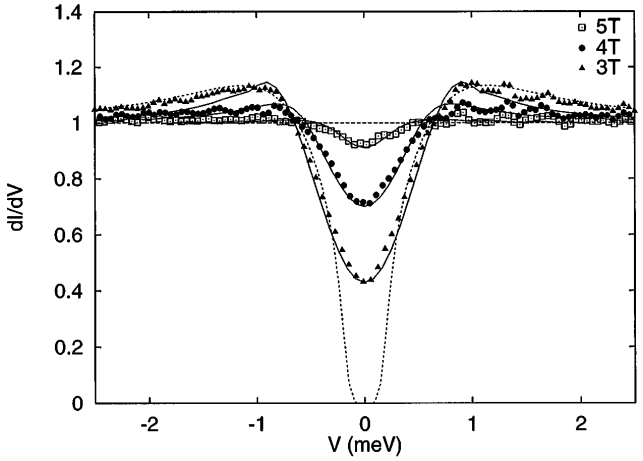

FIG. 1. Normalized junction conductance for PbBi/Ge with $T_c = 5.52$ K at 360 mK for several magnetic fields [18]. Solid lines: random matrix model ($\phi/W = 0.3,\ 0.8,\ 1.6$). Dashed line: Abrikosov-Gor'kov spectrum ($\zeta = 0.5$).

Second, a vortex solution in the Ginzburg-Landau model has a small "core" of size ξ, the coherence length, compared to λ, the penetration depth. Although the Ginzburg-Landau model does not contain electrons, the assumption is that the small core size over which the order parameter magnitude is reduced acts as a box in which electronic states are confined, leading to a small "normal fraction" of bound states. However, the order parameter does not act on electrons like a potential in a single-particle Schrödinger equation. The order parameter enters the Bogoliubov–de Gennes equations analogous to a spatially varying mass in a Dirac equation. Squaring the equation shows that currents also act as a confining potential. A more direct analogy might be made with a tornado: it is not just objects in the eye of the storm which are trapped and move with the tornado, but also objects in the circulating currents around the eye of the storm.

Bound states in the currents could explain why both the data and the random matrix model indicate the presence of more low energy states than expected from a Ginzburg-Landau picture. Solutions for the electronic structure around an isolated vortex would help address this question; initial numerical work shows clear deviation from Ginzburg-Landau behavior [22]. Note that if the bound states can be shown to extend out to a penetration depth, then the overlap with neighboring vortices, for fields above

H_{c1}, would cause a significant bandwidth. This would further increase the expected low energy density of states.

Experimentally, a surplus of low energy states is seen in the planar tunnel junction measurements [18] as well as in local scanning tunneling microscopy (STM) measurements in the vortex state [23] and near superconductor—normal-metal (SN) junctions [24]. Deviations from the linear in magnetic field heat capacity of the Ginzburg-Landau picture have been seen clearly in high-T_c superconductors [25], as well as conventional superconductors [26]. More systematic studies of the spectrum would help clarify the issue.

Another interesting feature of Eq. (15) is that there is a simple solution in the limit $W \ll \phi$. Physically, this corresponds to the situation where the energy scale for time-reversal symmetry breaking is small, yet many bare eigenstates are mixed chaotically. This might apply near SN junctions, in strongly anisotropic superconductors, or at magnetic fields much smaller than H_{c2}. (Note that it is necessary that many bare eigenstates be mixed, even if W is small, so that the large N limit for the matrix ensemble applies.) In this small-W limit, the cutoff function $h(x)$ becomes a delta function, and Eq. (15) is solved by

$$\hat{S}_1(\hat{E}, \hat{\varepsilon}) = (\hat{E} + \hat{\varepsilon})[1 - \sqrt{1 - (\hat{E}^2 - \hat{\varepsilon}^2)^{-1}}], \quad (17)$$

where the solution for σ_2 is given by $\sigma_2(E, \varepsilon) = \sigma_1(E, -\varepsilon)$ and all energies scale with ϕ: $\hat{E} = E/2\phi$, $\hat{\varepsilon} = \varepsilon/2\phi$, $\hat{S}_i = (2\phi)\sigma_i$. The resulting spectrum vanishes linearly near zero energy. This offers a possible alternative to the d-wave pairing explanation for the cuprate superconductors, for which many observations have indicated close to a linearly vanishing spectrum [27].

We also note that mesoscopic superconducting dots have recently been fabricated [28]. It is an interesting possibility that the matrix ensemble discussed here might describe the eigenvalue fluctuations of such systems, just as the standard Gaussian orthogonal ensemble describes the eigenvalue fluctuations of small metallic particles.

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