Experimental Evidence for Chaotic Scattering in a Fluid Wake

John C. Sommerer,* Hwar-Ching Ku, and Harold E. Gilreath
Milton S. Eisenhower Research & Technology Development Center,
The Johns Hopkins University Applied Physics Laboratory,
Laurel, Maryland 20723-6099
(Received 19 September 1996)

We present the first experimental evidence of chaotic scattering in a fluid wake. Measurements of tracer particles and dye in the stratified wake of a moving cylinder are shown to be consistent with four predictions based on simple models and direct numerical simulation: unstable periodic orbits were shadowed by tracer particles; streaklines marked by wake-delayed dye are shown to be fractal; early time-delay statistics of fluid elements interacting with the wake decayed exponentially; and finally, the fractal dimension of the wake is consistent with the dynamics of the wake, as measured by characteristic time delays and Lyapunov exponents. [S0031-9007(96)01924-2]

PACS numbers: 47.52.+j, 05.45.+b, 47.27.Vf, 83.50.Ws

Chaotic scattering [1–3] refers to the possibility that a system can manifest symptoms of chaos (such as sensitive dependence on initial conditions) for a finite time, even though the system's phase-space trajectory is asymptotically free during earlier and later epochs, i.e., the system is *open*. This can occur when the system's phase-space trajectory shadows a genuinely chaotic invariant set, which, being unstable, is not directly observable in a typical experiment. Chaotic scattering has been identified as a possible phenomenon in a wide variety of contexts, including celestial mechanics [4], microwave scattering [5], solar physics [6], geophysics [7], optics [8], atomic and nuclear physics [9], and fluid dynamics [10–14]. However, there have been very few [5] laboratory demonstrations of chaotic scattering, where sufficient control is available to test several independent predictions of a chaotic scattering model. In this Letter, we present experimental evidence for chaotic scattering in a fluid wake.

In a two-dimensional incompressible flow, there exists a time-dependent stream function such that a passive tracer's equations of motion have the exact form of Hamilton's equations. Thus the configuration-space paths of tracers in such a flow are also phase-space paths in an associated dynamical system. This identification has been used previously [10–13,15–17] to analyze several different fluid flows. If the associated dynamical system has chaotic scattering, then the flow will show the same properties in physical space; i.e., the scattering will be directly observable. References [12,13] suggested, based on direct numerical simulation of two-dimensional flow fields and on numerical investigation of analytical stream functions, that chaotic scattering should be observable in a two-dimensional fluid wake. Until now, that suggestion has not been realized in an actual experiment.

The most spectacular manifestation of chaotic scattering is that a generic scattering function exhibits an uncountable number of singularities, located on a fractal support in the space of impact parameters. In the case of a fluid wake, we take the *time delay* of fluid elements passing the perturbing body to be the scattering variable; almost all fluid elements eventually leave the wake and join free-streaming fluid elements, but some can be delayed for a long time. Unfortunately, exhibiting an actual scattering function is hardly practical for a real fluid wake, where accurate determination of the impact parameter of a fluid element exiting the wake region is currently impossible. Our demonstration of chaotic scattering is less direct, so we present several interlocking pieces of evidence consistent with the two-dimensional scattering model introduced in Refs. [12,13]. First, we observed that individual tracer particles shadowed periodic orbits predicted to exist in the closure of the unstable chaotic invariant set inhabiting the wake (here *wake* denotes the disturbed region of flow behind the moving body, where fluid element velocities differ from those in the free stream). Second, we observed that a wake-persistent fraction of an incident ensemble of tracers concentrated on a set that is approximately fractal. Third, we observed that the time delay statistics of fluid elements exiting the wake region showed an initial exponential decay. Fourth, and perhaps most conclusively, the fractal dimension of the set marked by the wake-persistent fluid elements is predicted by the time-delay statistics and local wake dynamics, according to a formula due to Kantz and Grassberger [18]. This last piece of evidence is analogous to using Lyapunov exponents to predict the dimensions of spatial fractal patterns via the Kaplan-Yorke formula [19]; this analysis has previously been accepted [16] as convincing evidence that some two-dimensional flows can be accurately described by low-dimensional dynamics.

Our experiments were conducted in JHU/APL's 1 m $\times$ 3 m $\times$ 8 m stratified flow facility, which allows experiments with arbitrary stable density profiles (see Fig. 1). We produced a thin (1–5 cm), strong density gradient at middepth, between layers of concentrated brine ($\rho \approx 1.2$ g cm^{-3}) on the bottom and fresh water on the top. We made all measurements within the thin mixing layer, where the Brunt-Väisälä frequency $N = [g\rho^{-1}|d\rho/dz|]^{1/2}$ was in the range $6 \leq N \leq 14$ s^{-1}.

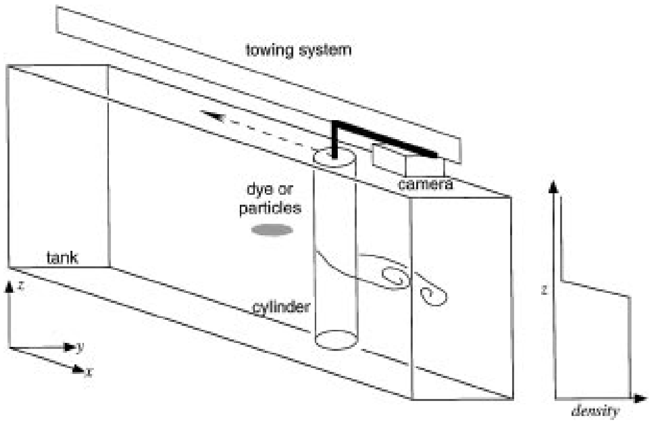

FIG. 1. Schematic diagram of experimental setup.

This very strong stratification effectively suppressed motion in the vertical direction, especially on time scales characteristic of fluid motion in the horizontal directions. The fluid wake was created by a cylinder (with vertical symmetry axis, radius $R_{\mathrm{cyl}} = 5$ cm) moving horizontally along the center of a channel between false walls (width $w = 20$ cm). This setup duplicated the geometry of Refs. [12,13]; however, the earlier numerical work considered flow around a stationary cylinder in a channel. With our moving cylinder we found some unimportant (i.e., not relevant to the issue of chaotic scattering) differences from the predictions of the earlier work, due to the different flow field near the channel walls. We towed the cylinder from an overhead track at low velocities U_{cyl} of a few $\mathrm{mm\,s^{-1}}$, yielding Reynolds numbers $\mathrm{Re} = 2R_{\mathrm{cyl}}U_{\mathrm{cyl}}/\nu$ in the range $100 \leq \mathrm{Re} \leq 250$. In this range of Reynolds number, after an initial transient, the velocity field in the wake region is (ideally) time periodic. This time dependence provides sufficient phase-space degrees of freedom ($x, y,$ and $t \bmod T_c$, where T_c is the period of the velocity field) to support chaos. The period T_c also provides a characteristic time scale against which to measure the dynamics (at $\mathrm{Re} = 100$, $T_c = 595$ s; at $\mathrm{Re} = 250, T_c = 205$ s). The dynamical system can be considered a three-dimensional continuous-time system, or alternatively, can be viewed as a two-dimensional area-preserving mapping, by considering snapshots taken at intervals of T_c.

The flow field in the wake is dominated by vortices which form behind the cylinder, alternating sides in the cross-stream direction. The nearby walls and viscosity quickly suppress this vorticity, so the entire von Karman street is reduced to only two vortices at any time.

The wake region was visualized from above using a CCD camera fixed with respect to the cylinder. Ultraviolet lamps inside the cylinder excited tiny fluorescent tracer particles (floating in the mixing layer [20]) or fluorescent dye, depending on the experiment.

In order to check that our experiment was indeed approximately two dimensional, with a time-periodic velocity field, we performed a strictly two-dimensional direct numerical simulation (DNS) of the flow field at the limiting Reynolds numbers. The DNS used a multigrid domain decomposition approach incorporating the pseudospectral element method [21]. We then compared computed *streaklines* with observed dye lines produced by a comb of outlets upstream from the cylinder. The correspondence was excellent (see Fig. 2), even at the lowest values of U_{cyl}, where background motions in the tank posed the largest threat to the periodicity of the velocity field, and where the initial, aperiodic transient was longest. Because background motions in the tank at the start of a run broke the flow symmetry, the initial transient was brief; comparison with the DNS indicates that unavoidable background motions were on the order of 0.01 U_{cyl}.

Periodic orbits.—The periodic orbits predicted [12,13] to inhabit the wake region are extremely unstable. In fact, the earlier numerical work used an analytical stream function that artificially expanded the boundary layer, making the periodic orbits more stable and allowing their numerical identification. Therefore we did not expect to see periodic orbits directly in our experiment. Our approach was to move the cylinder through a horizontal

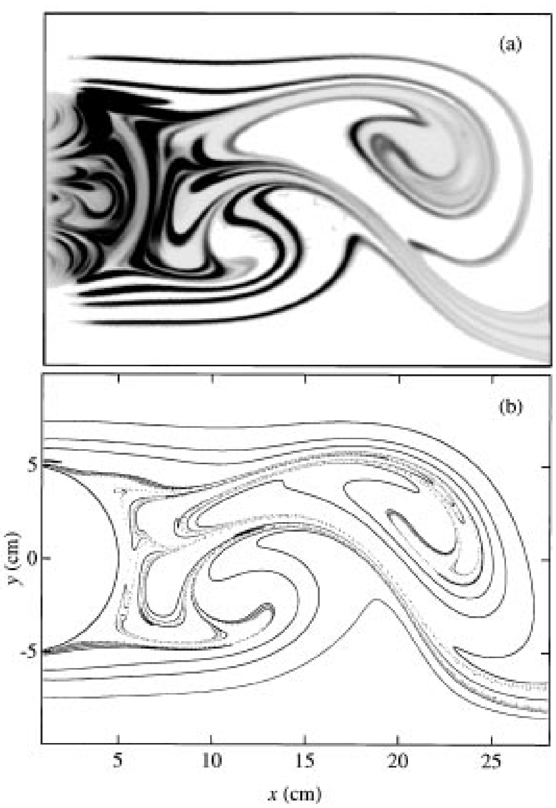

FIG. 2. Image of experimental flow field (a) shown above corresponding numerically computed streaklines (b) at $\mathrm{Re} = 100$. Cylinder is moving to the left. In (a), the thickness of dyed streaklines results from diffusion and slight three dimensionality of flow; extrusion of the dye under positive pressure results in some vertical extent to the dye lines.

sheet of tiny tracer particles and record the particle tracks with the CCD camera. By piecing together portions of the tracks of many different tracers, which shadowed the predicted orbits for various periods of time, we have identified several classes of predicted orbits. Of course, artificialities of the earlier numerical model resulted in detailed differences, but the generic character of the orbits is very similar to those shown in Refs. [12,13] (see Fig. 3).

Fractal nature of wake.—The unstable chaotic invariant set predicted [12,13] in the wake is a chaotic saddle C (topologically, a Cantor dust), having both an unstable manifold $\mathcal{W}^u$ and stable manifold $\mathcal{W}^s$. We expect $\mathcal{W}^u$, a foliation of curves, to have dimension $d(\mathcal{W}^u) = D_u + 1$, where $0 < D_u < 1$. When passive tracers impinge on the wake, those near $\mathcal{W}^s$ spend a long time shadowing C and tend to leave C along $\mathcal{W}^u$. Thus, in an experiment where the cylinder traverses a cross-stream stripe of dye, one expects that dye leaving the wake should trace out a good approximation of $\mathcal{W}^u$ and should also have dimension $d(\mathcal{W}^u)$. These arguments hold for capacity and information dimensions; the latter is more experimentally reliable [16] and more dynamically relevant [18,19].

For the Re $= 250$ case, the information dimension of the wake was measured (using the approach of Ref. [16]) to be $d(\mathcal{W}^u) = 1.3 \pm 0.1$. Because of the camera's limited resolution, the precision of this estimate is low, but the result is statistically bounded away from an integer.

Time delay statistics.—The picture presented above is complicated by the fact that fluid elements are delayed by *two* attributes of the system: the chaotic saddle C (a hyperbolic set) and the wall of the cylinder (a marginally stable, or parabolic set). Given the underlying Hamiltonian structure, there may also be Kolmogorov-Arnol'd-Moser (KAM) surfaces. Both latter possibilities would contaminate the exponentially distributed time-delay statistics expected of the hyperbolic component with algebraic decay at longer times.

We therefore considered the time delay of the earliest fluid elements leaving the wake. By moving the cylinder though a cross-stream stripe of dye, we marked an ensemble of impact parameters. The first dye to reach a strip $10R_{\text{cyl}}$ behind the cylinder (which was also behind the alternating vortices) had not interacted with the wake and was used to define the zero of time delay. We recorded the radiometric intensity of the dye that subsequently passed through the strip. The decay of the remaining dye (shown in Fig. 4) is initially exponential, indicating the predicted [12] interaction with a hyperbolic invariant set. Longer delays show a more complicated time dependence than the predicted simple t^{-2} algebraic decay. This discrepancy is likely due to overly simplistic approximation of the cylinder boundary layer in the analytical stream function of Ref. [12], but could also indicate the presence of KAM surfaces near the flow separation (although none were convincingly demonstrated in the experiment). In any case, the discrepancy is irrelevant to the argument for chaotic scattering.

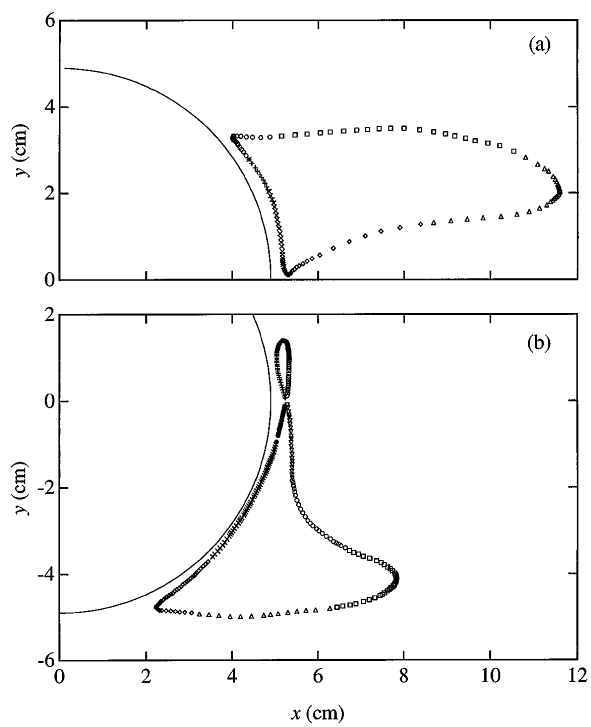

FIG. 3. Unstable periodic orbits in wake revealed by particle tracking. Particle coordinates were digitized and assembled into tracks. Portions of orbit shadowed by different particles are indicated by different plot symbols. (a) Period-1 orbit at Re $= 100$. (b) Period-1 orbit at Re $= 250$.

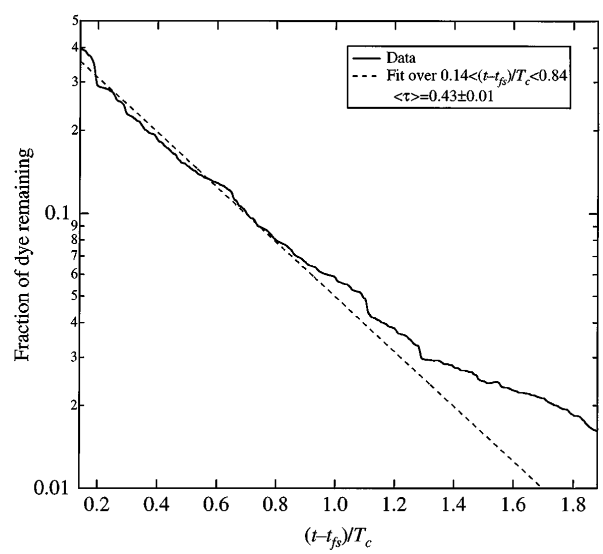

FIG. 4. Average time-delay statistics for dye interacting with wake. Since space of impact parameters includes the time at which fluid element encounters cylinder, relative to phase of periodic velocity field, an average is taken over eight runs with different upstream placement of the dye stripe.

Prediction of wake geometry.—Kantz and Grassberger [18] derived relationships among several quantities characteristic of the hyperbolic chaotic invariant set C of a two-dimensional mapping: the information dimensions of C and its stable and unstable manifolds [$d(C) = D_u + D_s$, $d(\mathcal{W}^s) = 1 + D_s$, and $d(\mathcal{W}^u) = 1 + D_u$], the Lyapunov exponents ($h_u > h_s$) along an invariant trajectory, and the characteristic delay time $\langle\tau\rangle$ of randomly initialized trajectories shadowing the invariant set:

$$D_s = 1 - \frac{1}{h_u\langle\tau\rangle}, \tag{1}$$

$$D_u = \frac{h_u - 1/\langle\tau\rangle}{|h_s|}. \tag{2}$$

Since for a conservative system $h_s = -h_u$, $D_u = D_s$.

In the previous section, we described the measurement of $\langle\tau\rangle$; for the Re $= 250$ case, $\langle\tau\rangle = 0.43 \pm 0.01$. The largest Lyapunov exponent was measured by particle tracking to be $h_u = 3.15 \pm 0.12$. Thus Eq. (2) predicts an information dimension $d(\mathcal{W}^u) = 1.26 \pm 0.03$, consistent with the experimental measurement of the wake information dimension.

In conclusion, we have presented four interlocking types of experimental evidence supporting the prediction [12,13] that chaotic scattering should describe the delay of fluid elements in a wake. This is a rare laboratory confirmation of chaotic scattering theory and an indication that chaotic scattering may have practical implications. Since the delay of fluid elements in a wake is one component of drag on a moving body, approximate calculation or even reduction of drag may be facilitated using simple chaotic scattering models. For example, because the invariants of the chaotic saddle in the wake may be expressed in terms of the properties of the unstable periodic orbits embedded in the saddle, disruption of the periodic orbits should affect the drag properties directly. Analysis of periodic orbits in simple models is much easier than direct numerical simulation of the Navier-Stokes equations.

We gratefully acknowledge experimental assistance from Joseph E. Hopkins and C. Howard Hoshall and useful discussions with E. M. Ziemniak. Much of the image processing was done using the public domain NIH Image program [22]. Our work was supported by the Department of the Navy, Space and Naval Warfare Systems Command.

*Electronic address: john.sommerer@jhuapl.edu

[1] B. Eckhardt, Physica (Amsterdam) **33D**, 89 (1988).
[2] P. Gaspard and S. A. Rice, J. Chem. Phys. **90**, 2225 (1989).
[3] E. Ott and T. Tél, Chaos **3**, 417 (1994).
[4] J. M. Petit and M. Hénon, Icarus **60**, 536 (1986); P. T. Boyd and S. L. W. McMillan, J. Info. Chaos **3**, 443 (1993).
[5] E. Doron, Phys. Rev. Lett. **65**, 3072 (1990).
[6] Y.-T. Lau and J. M. Finn, Astrophys. J. **366**, 577 (1991).
[7] J. McCloskey, C. J. Bean, and A. W. B. Jacob, Geophys. Res. Lett. **18**, 1901 (1991).
[8] J. H. Jensen, J. Opt. Soc. Am. A **10**, 1204 (1993).
[9] A. Rapisarda and M. Baldo, Phys. Rev. Lett. **66**, 2581 (1991); Y. Gu and J.-M. Yuan, Phys. Rev. A **47**, R2442 (1993).
[10] B. Eckhardt and H. Aref, Philos. Trans. R. Soc. London A **326**, 655 (1988).
[11] T. H. Solomon, E. R. Weeks, and H. L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
[12] C. Jung and E. Ziemniak, J. Phys. A **25**, 3929 (1992); C. Jung, T. Tél, and E. Ziemniak, Chaos **3**, 555 (1993); T. Tél, E. M. Ziemniak, and C. Jung, Physica (Amsterdam) **76D**, 123 (1994).
[13] E. M. Ziemniak and C. Jung, Phys. Lett. A **202**, 263 (1995).
[14] G. Stolovitzky, T. J. Kaper, and L. Sirovich, Chaos **5**, 671 (1995).
[15] R. P. Behringer, S. D. Meyers, and H. L. Swinney, Phys. Fluids **A3**, 1243 (1991).
[16] J. C. Sommerer and E. Ott, Science **259**, 335 (1993); J. C. Sommerer, Physica (Amsterdam) **76D**, 85 (1994).
[17] T. M. Antonsen, Jr. *et al.*, Phys. Rev. Lett. **75**, 3438 (1995); J. C. Sommerer, Phys. Fluids **8**, 2441 (1996).
[18] H. Kantz and P. Grassberger, Physica (Amsterdam) **17D**, 75 (1985).
[19] J. Kaplan and J. A. Yorke, in *Lecture Notes in Mathematics*, edited by H. O. Peitgen and H. O. Walther (Springer-Verlag, Berlin, 1978), Vol. 730, p. 228.
[20] M. S. Pervez and T. H. Solomon, Experiments in Fluids **17**, 135 (1994).
[21] H.-C. Ku, R. S. Hirsh, T. D. Taylor, and A. P. Rosenburg, J. Comput. Phys. **83**, 260 (1989); H.-C. Ku, A. P. Rosenburg, and T. D. Taylor, in *Lecture Notes in Physics*, edited by K. W. Morton (Springer-Verlag, Berlin, 1990), Vol. 371, p. 223; H.-C. Ku, H. E. Gilreath, R. Raul, and J. C. Sommerer, Contemp. Math. **180**, 509 (1994).
[22] W. Rasband, U.S. National Institutes of Health, available from Internet by anonymous ftp from zippy.nimh.nih.gov or on floppy disc from NTIS, 5285 Port Royal Rd., Springfield, VA 22161, part number PB93-504868.