Noise-Induced Riddling in Chaotic Systems

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Recent works have considered the situation of riddling where, when a chaotic attractor lying in an invariant subspace is *transversely stable*, the basin of the attractor can be riddled with holes that belong to the basin of another attractor. We show that riddling can be induced by arbitrarily small random noise *even if the attractor is transversely unstable*, and we obtain universal scaling laws for noise-induced riddling. Our results imply that the phenomenon of riddling can be more prevalent than expected before, as noise is practically inevitable in dynamical systems. [S0031-9007(96)01791-7]

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The discovery of the phenomenon of riddled basins [1,2] brings another important study area to the forefront of the research in chaotic dynamics. Riddling usually occurs in chaotic systems with symmetric invariant subspaces. When there is a chaotic attractor in the invariant subspace and another attractor (say, nonchaotic) off the invariant subspace, if the chaotic attractor is stable with respect to transverse perturbations, the basin of the chaotic attractor can be riddled with holes belonging to the basin of the attractor that is off the invariant subspace [1]. Recent work demonstrated that the onset of riddling is typically induced by the loss of the transverse stability of some low-period periodic orbit embedded in the chaotic attractor [3]. As a system parameter changes further, blowout bifurcation can occur in which typical trajectories on the *whole* chaotic attractor becomes transversely unstable [2,4]. After the blowout bifurcation, riddling of the chaotic attractor in the invariant subspace disappears.

In this paper, we present analysis and numerical results which demonstrate that even when the chaotic attractor in the invariant subspace is transversely unstable, if there are coexisting attractors symmetrically located off the invariant subspace, riddling in the basins of these attractors can still occur when there is small-amplitude random noise present. We call this type of riddling the *noise-induced riddling.* In particular, let p be a system parameter, p_c be the blowout bifurcation point, and **S** denote the invariant subspace. Assume there are two attractors, denoted by A and B, one above and another below **S**. When noise is absent, for $p \lesssim p_c$ there are two Cantor-like sets (closed) of positive Lebesgue measure in the phase space, one above and another below **S**, that are transversely stable. Points in the sets are attracted towards **S** and, hence, they belong to the basin of the chaotic attractor in **S**. Since the Cantor-like sets are closed and have positive measure, the basin of the chaotic attractor in **S** is riddled. The complement of these two closed sets are two open sets that belong to the basins of the attractors A and B, respectively. This situation is shown schematically in Fig. 1. For $p \gtrsim p_c$, the Cantor-like sets are still stable, but they have now Lebesgue measure zero and, hence, **S** is transversely unstable. In this case, the chaotic attractor in **S** becomes a repeller in the transverse direction, and trajectories above (below) **S** are repelled away from **S** and are eventually attracted to A (B). The entire phase-space regions above and below **S** are the basins of attraction for typical trajectories to the attractors A and B, respectively, and there is no riddling in this case. When small noise is present, the Cantor-like sets become "fattened" in the phase space. For both p below p_c and p above p_c, trajectories can come close to **S** due to the transversely stable Cantor-like sets, and there is a nonzero probability that trajectories above **S** can be kicked across **S** and be attracted below towards B due to noise, as shown in Fig. 1. The initial conditions in the fattened Cantor-like set above **S** are thus in the basin of B (the noise-induced basin of B) and form a riddled structure. By symmetry, the basin of A below **S** is also riddled. We emphasize that riddling of the attractors off **S** occurs on both sides of p_c, but riddling is observable only at scales larger than the noise amplitude.

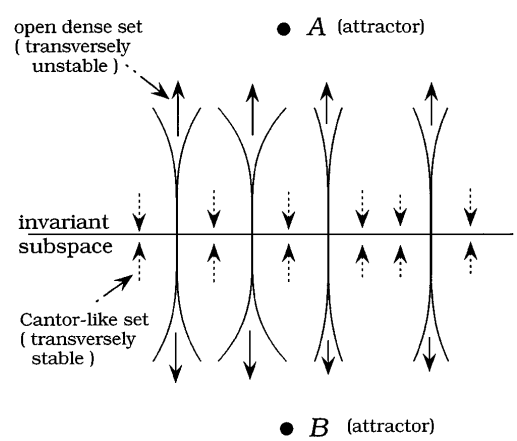

FIG. 1. A chematic illustration of two invariant sets in the phase space for p around the blowout bifurcation point p_c. One is open dense and transversely unstable; another is transversely stable but closed. The two symmetric closed sets above and below **S** correspond to the noise-induced basins.

In the following, we first present numerical evidence illustrating the phenomenon of noise-induced riddling. We then consider an analyzable model which can be solved by employing the diffusion approximation. We derive universal scaling laws associated with the noise-induced riddling. *The main implication of our result is that the phenomenon of riddling may be more prevalent than expected before, as noise is inevitable in practical situations.*

We consider the following general class of dynamical systems,

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}(\mathbf{x}_n) + \text{high order terms } of\ \mathbf{y}_n\,, \\ \mathbf{y}_{n+1} &= g(\mathbf{x}_n, p)\mathbf{y}_n + \text{high order odd terms } of\ \mathbf{y}_n\,, \end{aligned} \tag{1}$$

where $\mathbf{x} \in R^{N_S}$ ($N_S \geq 1$), $\mathbf{y} \in R^{N_T}$ ($N_T \geq 1$), $\mathbf{f}(\mathbf{x}_n)$ is a map that has a chaotic attractor, $g(\mathbf{x}_n, p)$ is a scalar function, and p is the bifurcation parameter. The invariant subspace is defined by $\mathbf{y} = 0$ because for initial $\mathbf{y}_0 = 0$, trajectories have $\mathbf{y}_n = 0$ for all times. The largest transverse Lyapunov exponent is given by $\Lambda_\perp = \lim_{M\to\infty}(1/M)\sum_{n=1}^{M} \ln|\partial\mathbf{y}_{n+1}/\partial\mathbf{y}_n|_{\mathbf{y}_n=0} \cdot \mathbf{u}|$, where $\mathbf{u}$ is a random unit vector in the transverse subspace. Since p_c is the blowout bifurcation point, we have $\Lambda_\perp > 0$ for $p > p_c$. Our main goal is to understand how noise can induce riddling. To illustrate our findings, we consider the following two-dimensional version of Eq. (1),

$$\begin{aligned} x_{n+1} &= f(x_n) + q y_n^2\,, \\ y_{n+1} &= p x_n y_n + y_n^3 + \epsilon\sigma_n\,, \end{aligned} \tag{2}$$

where $\sigma_n \in [-1, 1]$ is a random number, $\epsilon \ll 1$ is the noise amplitude, $p > 0$, and q is another parameter. In Eq. (2), both the invariant subspace ($y = 0$) and the transverse subspace are one-dimensional. Note that noise only affects the dynamics in the vicinity of the invariant subspace $y = 0$, as the noise term in Eq. (2) is negligible when $|y|$ is large. We choose $f(x_n)$ to be the doubling transformation $2x\ \mathrm{mod}(1)$ that produces a chaotic attractor with uniform invariant density for $x \in [0, 1]$. In this case, the transverse Lyapunov exponent is given by $\Lambda_\perp = \int \ln|px|\rho(x)\,dx = \int_0^1 \ln|px|\,dx = \ln p - 1$. The blowout bifurcation point is $p_c = e = 2.71828\ldots$. From the second equation in Eq. (2), we see that for $p > 0$, if $|y_n| > 1$, then $|y_{n+1}| > |y_n|$. Thus $y = \pm\infty$ are the two attractors located symmetrically with respect to $y = 0$. For $\epsilon = 0$ and $p \lesssim p_c$, the chaotic attractor of the doubling transformation at $y = 0$ is also an attractor of the full phase space, the basin of which is riddled with holes belonging to the basins of the $y = \pm\infty$. For $p \gtrsim p_c$, the $y = 0$ chaotic attractor is no longer a global attractor. In this case, $y = \pm\infty$ are the only global attractors of the system. If $\epsilon = 0$, y_n cannot change sign, and, consequently, the basins of the $y = +\infty$ and $y = -\infty$ attractors are $y > 0$ and $y < 0$, respectively, and the basin boundary is the one-dimensional line $y = 0$ for $p \gtrsim p_c$.

We now argue that for $p \gtrsim p_c$, noise can induce riddling between the basins of the $y = +\infty$ and $y = -\infty$ attractors. We first note that since $\Lambda_\perp$ is only slightly positive, there is a set of Lebesgue measure zero points embedded in the $y = 0$ chaotic attractor that are transversely stable. Although typical trajectories asymptote to $y = \pm\infty$ eventually, usually they can spend a long time in the vicinity of $y = 0$ before doing so. Imagine we turn on the noise now. Because of noise, an infinite number of channels open at the set of transversely stable points, allowing trajectories to pass through $y = 0$. There is now a nonzero probability that trajectories coming close to $y = 0$ from the positive side can tunnel through $y = 0$ to the negative side and asymptote to the $y = -\infty$ attractor, and vice versa. Thus, as long as there is noise, no matter how small, there are points with $y > 0$ ($y < 0$) that belong to the basin of the $y = -\infty$ ($y = +\infty$) attractor. Since the noise-induced basins correspond to the transversely stable closed sets above and below $y = 0$, these basins must be riddled. That is, for any point with $y > 0$ ($y < 0$) that goes to the $y = -\infty$ ($y = +\infty$) attractor, there are points arbitrarily nearby (down to the noise scale) that go to the $y = +\infty$ ($y = -\infty$) attractor. Figure 2 shows the basin of the $y = -\infty$ attractor in $y > 0$ (black dots), where the parameter setting is $p = 2.8 > p_c$, $q = 0.1$, and $\epsilon = 10^{-12}$. To produce this figure, a grid of 2048×2048 of initial conditions is chosen in the region $0 \leq x \leq 1$ and $0 < y < 1$. If an initial condition has $y < -10^3$ within 10^7 iterations, a black dot is plotted at the location of the initial condition. Otherwise we leave it blank. The figure exhibits typical features of a riddled basin [1,2].

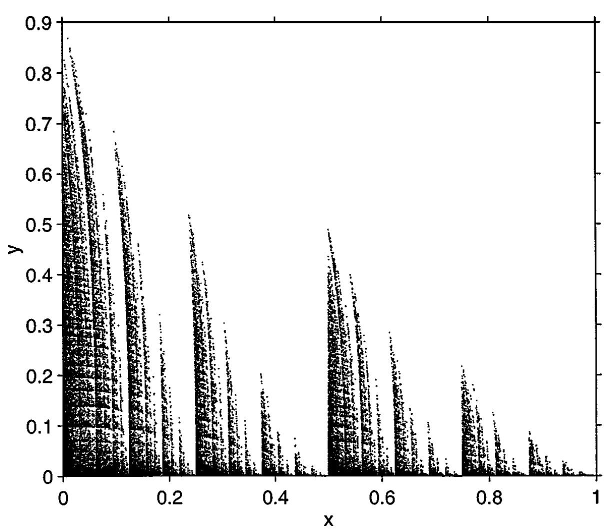

FIG. 2. For Eq. (2), the noise-induced basin of the $y = -\infty$ attractor in the $y > 0$ half plane. The parameter setting is $a = 0.1$, $p = 2.8 > p_c$ ($\Lambda_\perp \approx 0.0296$), and $\epsilon = 10^{-12}$ (the noise amplitude). Without noise, at this parameter setting the $y > 0$ half plane is the basin of the $y = +\infty$ attractor except a set of Lebesgue measure zero.

To characterize noise-induced riddling, we first compute the fraction of points $f_-(\epsilon, y_0)$ on a fixed line $y_0 \gtrsim 0$ that belong to the basin of the $y = -\infty$ attractor as ϵ changes. Figure 3(a) shows $\log_{10} f_-(\epsilon, y_0)$ versus $\log_{10}\epsilon$ for $p = 2.8$, $q = 0.1$, and $10^{-12} < \epsilon \leq 10^{-6}$, where 10^6 initial conditions are chosen on the line $y = 0.01$ to compute $f_-(\epsilon, y_0)$. We see that the plot can be roughly fitted by a straight line, indicating an algebraic scaling relation between $f_-(\epsilon, y_0)$ and ϵ: $f_-(\epsilon, y_0) \sim \epsilon^{\alpha}$, where $\alpha > 0$ is the algebraic scaling exponent. In Fig. 3(a), the exponent is $\alpha \approx 0.050$. Next, we compute, for a fixed noise amplitude ϵ, a fraction of initial conditions $f_-(\epsilon, y_0)$ that asymptote to the $y = -\infty$ attractor change as y_0 ($y_0 \gtrsim 0$) increases. Figure 3(b) shows $\log_{10} f_-(\epsilon, y_0)$ versus $\log_{10} y_0$ for $10^{-12} < y_0 \leq 10^{-6}$, where $\epsilon = 10^{-12}$ is fixed, and 10^6 initial conditions are used to compute $f_-(\epsilon, y_0)$ for each y_0. We also obtain an algebraic scaling relation, $f_-(\epsilon, y_0) \sim y_0^{-\beta}$, where $\beta > 0$ is the scaling exponent. In Fig. 3(b), the exponent is $\beta \approx 0.065$. We see that α and β have similar values.

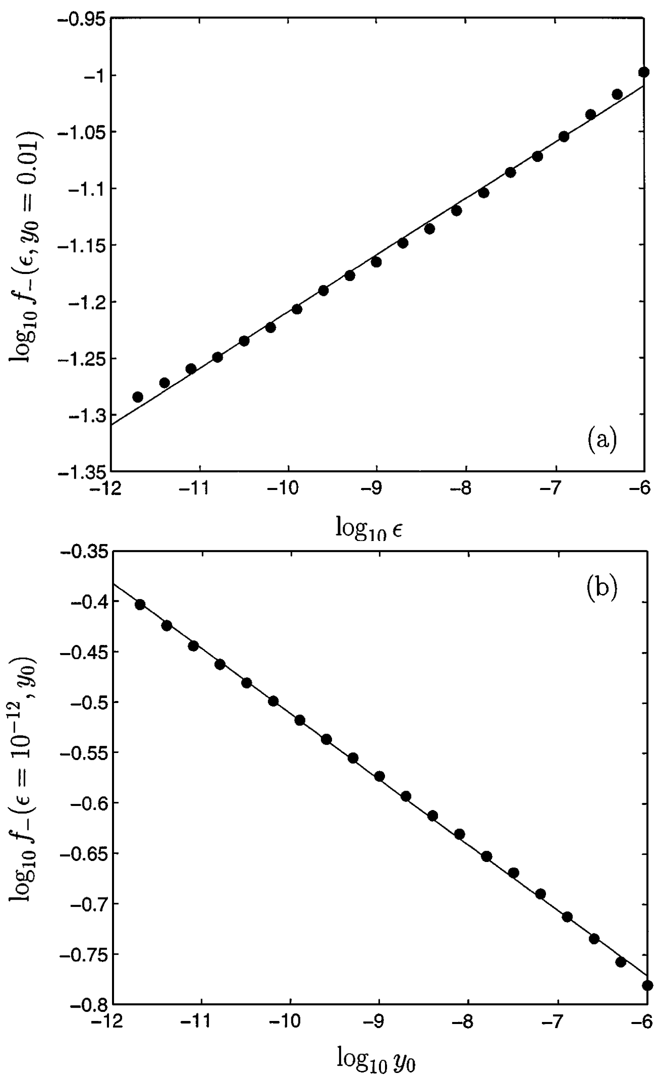

FIG. 3. (a) At $y_0 = 0.01$, on a logarithmic scale, the probability $f_-(\epsilon, y_0)$ that a random x_0 asymptotes to the $y = -\infty$ attractor versus the noise amplitude ϵ. The plot indicates that roughly, $f_-(\epsilon, y_0) \sim \epsilon^{0.05}$. Other parameters are $a = 0.1$ and $p = 2.8$. (b) At $\epsilon = 10^{-12}$, $f_-(\epsilon, y_0)$ versus y_0 on a logarithmic scale. Roughly, we have $f_-(\epsilon, y_0) \sim y_0^{-0.065}$.

To understand the scaling of the noise-induced riddling, we consider an analyzable model with additive noise. The model is a two-dimensional map defined in the region $0 \leq x \leq 1$ and $-\infty < y < \infty$, as follows:

$$x_{n+1} = \begin{cases} (1/a)x_n, & \text{for } x_n < a, \\ (1/b)(x_n - a), & \text{for } x_n > a, \end{cases}$$
$$y_{n+1} = \begin{cases} cy_n + \epsilon\sigma_n, & \text{for } x_n < a, \\ dy_n + \epsilon\sigma_n, & \text{for } x_n > a, \end{cases} \tag{3}$$

where $0 < a < 1$, $b = 1 - a, c > 1, 0 < d < 1$, and $\epsilon\sigma_n$ is the small noise term similar to that in Eq. (2). The invariant subspace is $y = 0$ in which there is a chaotic attractor with the Lyapunov exponent $\Lambda_x = a\ln(1/a) + b\ln(1/b) > 0$. The y dynamics involves both expansion and contraction, and there are two attractors located at $y = \pm\infty$, respectively. The transverse Lyapunov exponent is $\Lambda_\perp = a\ln c + b\ln d$. Thus, $\Lambda_\perp \geq 0$ for $a \geq a_c$ and $\Lambda_\perp < 0$ for $a < a_c$, where $a_c = |\ln d|/(\ln c + |\ln d|)$. For $a > a_c$ and $\epsilon = 0$, except for a set of measure zero, the upper half plane ($y > 0$) and the lower half plane ($y < 0$) are the basins of the $y = +\infty$ and $y = -\infty$ attractors, respectively. Concentrating on the $y > 0$ half plane and defining $Y_n \equiv -\ln y_n$, in the noise-free case we obtain a random walk in terms of Y_n for the y dynamics, $Y_{n+1} = \gamma_n + Y_n$, where $\gamma_n = \overline{c} \equiv -\ln c < 0$ with probability a and $\gamma_n = \overline{d} \equiv -\ln d > 0$ with probability $b = 1 - a$. We are interested in the case where $a \gtrsim a_c$ so that $\Lambda_\perp \gtrsim 0$. In this case, on average the trajectory moves slowly in the y direction, and, hence, the random walk can be solved by using the diffusion approximation. Let $P(Y, Y_0, n)$ be the probability distribution function for Y (given that x_0 is chosen randomly on the horizontal line segment $y = y_0, 0 \leq x_0 \leq 1$), and we obtain the following diffusion equation for $P(Y, Y_0, n)$ [5]:

$$\frac{\partial P}{\partial n} + \nu \frac{\partial P}{\partial Y} = D \frac{\partial^2 P}{\partial Y^2}, \tag{4}$$

where $\nu = a\overline{c} + b\overline{d} = -\Lambda_\perp$ is the average drift, and $D \equiv \frac{1}{2}\langle(\delta Y - \langle\delta Y\rangle)^2\rangle = \frac{1}{2}ab(\overline{c} - \overline{d})^2$ is the diffusion coefficient (the average $\langle\cdots\rangle$ is with respect to initial random values of x_0). For $a \gtrsim a_c$ we see that $\nu \lesssim 0$, indicating that Y gradually approaches $-\infty$ (or $y \to \infty$). Assuming that all initial conditions start from y_0, where $0 < y_0 < 1$ (or $Y_0 > 0$), we have the following initial condition for Eq. (4): $P(Y, Y_0, 0) = \delta(Y - Y_0)$. To model the effect of noise, we note that once a trajectory falls within distance ϵ of $y = 0$, it can tunnel through $y = 0$ and asymptotes to $y = -\infty$. Roughly speaking, there is an absorbing boundary for the random walker at $\overline{\epsilon} \equiv \ln(1/\epsilon) > 0$. As a crude approximation, we have the following boundary condition: $P(\bar{\epsilon}, Y, n) = 0$. The diffusion equation (4), together with the above initial and boundary conditions, can be solved by using the standard Laplace-transformation method [5]. Letting $\overline{P}(Y, Y_0, s) \equiv \int_0^\infty P(Y, Y_0, n)e^{-sn}\, dn$ be the Laplace transform of $P(Y, Y_0, n)$, we obtain

$$D\,\frac{d^2\overline{P}(Y, Y_0, s)}{dY^2} + \Lambda_\perp \frac{d\overline{P}(Y, Y_0, s)}{dY} - s\overline{P}(Y, Y_0, s) = -\delta(Y - Y_0)\,.$$

With the boundary condition $\overline{P}(\bar{\epsilon}, Y_0, s) = 0$, we obtain the solution $\overline{P}(Y, Y_0, s) = C_1 e^{\lambda_1 Y} + C_2 e^{\lambda_2 Y}$ for $Y > Y_0$, and $\overline{P}(Y, Y_0, s) = C_3 e^{\lambda_1 Y}$ for $Y < Y_0$ where $\lambda_1 = \frac{1}{2}\eta(\Delta - 1)$, $\lambda_2 = \frac{1}{2}\eta(\Delta + 1)$, $\eta \equiv \Lambda_\perp/D > 0$, and $\Delta \equiv \sqrt{1 + 4Ds/(\Lambda_\perp^2)}$. The coefficients are, $C_2 = [1/D(\lambda_1 - \lambda_2)]\exp(-\lambda_2 Y_0)$, $C_1 = -C_2 \exp[(\lambda_2 - \lambda_1)\bar{\epsilon}]$, and $C_3 = C_2\{\exp[(\lambda_2 - \lambda_1)Y_0] - \exp[(\lambda_2 - \lambda_1)\bar{\epsilon}]\}$.

We can now calculate the scaling. Let $F_+(n)$ be the probability that the walker has not reached within ϵ of $y = 0$ at time n. The Laplace transform of $F_+(n)$ is given by $\overline{F}_+(s) = \int_{-\infty}^{\bar{\epsilon}} \overline{P}(Y, Y_0, s)\, dY$. Thus, we have $\overline{F}_+(s) = 1/s - (1/s)\exp[-\lambda_2(Y_0 - \bar{\epsilon})]$. Performing the inverse-Laplace transform by noting that there are a pole at $s = 0$ and a branch singularity at $s = s^* \equiv \Lambda_\perp^2/4D > 0$, we obtain

$$\begin{aligned} F_+(n) = 1 &- \exp[-\lambda_2(s = 0)\,(Y_0 - \bar{\epsilon})] \\ &- \frac{1}{s^*}\exp[-\lambda_2(s = s^*)\,(Y_0 - \bar{\epsilon})] \\ &\times \exp(-s^* n)\,. \end{aligned} \tag{5}$$

In the limit $n \to \infty$, $F_+(n)$ is the probability that the random walker has never reached $Y \geq \bar{\epsilon}(y \leq \epsilon)$ and, hence, $F_+(\infty)$ is the fraction of the $y = +\infty$ basin in the upper half plane. Therefore, the noise-induced fraction of points at $y_0 > 0$ that belong to the $y = -\infty$ basin is given by $f_-(\epsilon, y_0) = 1 - \lim_{n\to\infty} F_+(n) = \exp[\eta(Y_0 - \bar{\epsilon})]$. Finally, we obtain the following algebraic scaling relation:

$$f_-(\epsilon, y_0) \sim \epsilon^\alpha y_0^{-\beta}, \tag{6}$$

where the scaling exponents are given by [6] $\alpha = \beta = \eta = \Lambda_\perp/D$. Because of symmetry, the same scaling holds for the fraction of the $y = +\infty$ basin in the lower half plane $y < 0$. Since the scaling exponents α and β only depend on $\Lambda_\perp = -\nu$ and D, which are the two fundamental parameters in the diffusion equation, we expect the scaling Eq. (6) to hold *universally* for noise-induced riddling in the parameter regime where the diffusion approximation is valid, regardless of the details of the system.

To check the universality of the scaling relation Eq. (5), we note that in our numerical model Eq. (2), the transverse Lyapunov exponent and the diffusion coefficient are given by $\Lambda_\perp = \ln p - 1$ and $D = \frac{1}{2}\int[\ln(px) - \Lambda_\perp]^2 \times \rho(x)\, dx = \frac{1}{2}$, respectively. Thus we have $\eta = 2(\ln p - 1)$ for $p \gtrsim p_c$. For $p = 2.8$, we have $\eta \approx 0.059$. This agrees fairly well with the numerical values of $\alpha \approx 0.050$ and $\beta \approx 0.065$ in Figs. 3(a)–3(b).

Chaotic dynamical systems with invariant symmetric subspaces are quite common. The coexistence of various attractors is also common. When there is a chaotic attractor in the invariant subspace, previous works have firmly established the phenomenon of riddling for parameter regimes *only* below the blowout bifurcation point where the chaotic attractor is transversely stable. The results of this paper show that even beyond the blowout bifurcation, riddling can still occur when there is arbitrarily small noise present. Thus, in different forms, riddling can occur in wide parameter regimes about the blowout bifurcation point. The universal scaling behaviors associated with the noise-induced riddling have been obtained in this paper. Since noise is inevitable in real dynamical systems, we expect riddling to occur commonly in dynamical systems with symmetry.

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[6] The equality of the scaling exponents α and β can also be seen via a dimension analysis. Since ϵ and y_0 have the same physical dimensions (distances), while $f_-(\epsilon, y_0)$ is just a number, one must have $\alpha = \beta$ from Eq. (6).