Susceptibility of Chaotic Systems to Perturbations

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(Received 25 January 1996)

The susceptibility of chaotic Hamiltonian systems with 2 degrees of freedom to a perturbation of harmonical time dependence is studied. The dispersion relation for the susceptibility that includes the Lyapunov exponent is established. The equations that connect the parameters of the susceptibility with those of the power spectrum of the coordinate and the density of states are derived from the equivalence of quantum and classical susceptibilities in the limit $\hbar \to 0$. The susceptibility of the Pullen-Edmonds nonlinear oscillator is calculated as an example. [S0031-9007(96)01840-6]

PACS numbers: 05.45.+b, 03.65.Sq

The susceptibility of a dynamic system to a small periodic perturbation is one of its basic characteristics. If the perturbation is due to an applied alternating electric field, the susceptibility is proportional to the polarizability of the system, which is a fundamental quantity in the theory of interaction of radiation with matter. Further, the susceptibility can be related to the internal dynamics of the system. The best known example is the fluctuation-dissipation theorem by Callen and Welton [1] that relates the averaged square value of the fluctuating dynamic variable to the imaginary part of the susceptibility of the system at thermal equilibrium.

In this paper we study the general properties of the susceptibility of autonomous Hamiltonian chaotic systems with 2 degrees of freedom and a given energy E in chaotic dynamics [2].

Let us consider a model of quantum systems with 2 degrees of freedom with the Hamiltonian operator

$$\hat{H}(\hat{p}, \hat{r}) = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + U(\hat{x}, \hat{y}), \tag{1}$$

where $\hat{x}, \hat{y}$ are the operators of Cartesian coordinates, $\hat{p}_x, \hat{p}_y$ are the Cartesian components of the momentum, and m is the mass of the particle. The perturbation is described by adding to $\hat{H}$ the operator

$$\hat{V}(\hat{x}, t) = -\hat{x} f_0 \cos \omega t . \tag{2}$$

If this perturbation is switched on adiabatically, the response $\langle \hat{x}(t) \rangle$ to the perturbation $\hat{V}(\hat{x}, t)$ (with the infinitesimal imaginary quantity added to the frequency) has the form

$$\langle \hat{x}(t) \rangle = f_0[\alpha'(\omega) \cos \omega t + \alpha''(\omega) \sin \omega t] \tag{3}$$

with usual assumption that $\langle \hat{x} \rangle = 0$ for the unperturbed system [1].

This equation defines the real (α') and imaginary (α'') parts of the generalized susceptibility [1]. The angular brackets in Eq. (3) denote averaging over the initial quantum state. If the system $\hat{H}$ has a discrete energy spectrum E_i, the susceptibility of the system in some stationary state $|n\rangle$ is given by

$$\alpha(\omega) = \frac{1}{\hbar} \sum_k \frac{2|x_{nk}|^2 \omega_{kn}}{\omega_{kn}^2 - (\omega + i0)^2}, \tag{4}$$

where x_{nk} is the matrix element of the coordinate, and $\omega_{kn} = (E_k - E_n)/\hbar$ is the transition frequency between the states $|k\rangle$ and $|n\rangle$.

If the classical model with an unperturbed Hamiltonian function corresponding to (1) is chaotic, then one would expect the chaos to show up in the features of the quantum model in the quasiclassical limit [3,4]. In this case the expression (4) is practically useless. Because of the weakness of selection rules in quantum chaotic systems, the main contribution in the sum over k will be given by a large (and increasing with $\hbar \to 0$) number of terms irregularly depending on k.

It is natural to describe the properties of the quantum system defined by Eqs. (1) and (2) in the limit $\hbar \to 0$ by classical susceptibility. This quantity can be defined by calculating the linear response of the system $\langle x(t) \rangle$, where $x(t)$ is the solution of canonical equations of motion that are linearized in the vicinity of the phase trajectory $\{p(t), r(t)\}$. The angular brackets in the classical case stand for averaging over the initial conditions. The classical susceptibility of nonlinear regular systems with 1 degree of freedom has been treated long ago [5].

For the chaotic motion, among the fundamental set of the solutions of the system of linearized equations of motion there is at least one that increases exponentially with time: $\varphi_1(t) \sim \exp \sigma t$, where $\sigma > 0$ is the (maximal) Lyapunov exponent. This growth indicates the existence of a pole of the function $\alpha(\omega)$ at the point $\omega = i\sigma$ in the upper half-plane of complex frequency. The presence of such a pole in the susceptibility of unstable systems is well known [6]. The causality condition for the temporal behavior of the response of the system is fulfilled if the integration contour in the plane of complex ω passes above this pole. The symmetry condition $\alpha(-\omega) = \alpha^*(\omega)$ gives another pole at the point $\omega = -i\sigma$. Therefore the susceptibility of the chaotic system can be put in the form

$$\alpha(\omega) = \frac{\gamma}{\omega^2 + \sigma^2} + \beta(\omega), \tag{5}$$

where γ is some constant and $\beta(\omega)$ has no singularities in the upper half-plane. Since $\beta(\omega)$ obeys the usual Kramers-Kronig dispersion relation [1], the dispersion relation for $\alpha(\omega)$ can be written as

$$\alpha(\omega) = \frac{2}{\pi} \int_0^\infty \frac{\xi \alpha''(\xi)}{\xi^2 - (\omega + i0)^2} d\xi + \frac{\gamma}{\omega^2 + \sigma^2}. \tag{6}$$

This equality holds for any classical dynamic chaotic system with only one positive Lyapunov exponent.

In the high frequency range $\omega \to \infty$ both parts of this equality can be expanded in series in powers of ω^{-2}. To determine the coefficients of this expansion of the left-hand side (lhs) for the Hamiltonian system (1), the quantum expression (4) can be used. By equating the coefficients at ω^{-2} and ω^{-4} the following equations can be obtained with the help of the well known sum rules [7]:

$$\frac{1}{m} = \frac{2}{\pi} \int_0^\infty \xi \alpha''(\xi)\, d\xi - \gamma, \tag{7}$$

$$\frac{1}{m^2} \langle U_{xx} \rangle = \frac{2}{\pi} \int_0^\infty \xi^3 \alpha''(\xi)\, d\xi + \gamma\sigma^2, \tag{8}$$

where $U_{xx} = \partial^2 U / \partial x^2$. If the chaotic motion is ergodic within a given component $C(E)$ on the energy surface $H = E$, the matrix element of the second derivative of the potential U_{xx} in the lhs of Eq. (8) can be replaced by its average over $C(E)$. In our notation Eq. (8) retains its form.

The rate of energy absorption Q by the perturbed system is proportional to the imaginary part of its susceptibility [1]:

$$Q = \frac{\omega}{2} \alpha''(\omega) f_0^2. \tag{9}$$

This quantity can be calculated in the quantum model by treating the energy spectrum as a quasicontinuous one. Since the perturbing force f_0 is arbitrarily small, the absorption rate is given by the Fermi golden rule:

$$Q = \hbar\omega(\dot{W}_+ - \dot{W}_-). \tag{10}$$

$\dot{W}_+$ and $\dot{W}_-$ in Eq. (10) are transition rates from the initial state $|n\rangle$ accompanied by the absorption $(+)$ or emission $(-)$ of energy quanta $\hbar\omega$:

$$\dot{W}_\pm = \frac{2\pi}{\hbar} \frac{f_0^2}{4} |x_{nk}|^2 \rho(E_{k\pm}), \tag{11}$$

where $\rho(E_{k\pm})$ is the density of states near the final state with the energy $E_{k\pm} = E_n \pm \hbar\omega$. Matrix elements x_{nk} of quantum chaotic systems strongly fluctuate with variation of k [3,4]. However, the averaged squared quantity $\overline{|x_{nk}|^2}$ in the limit $\hbar \to 0$ is smooth. Because of the correspondence principle it is proportional to the power spectrum $S_x(E, \omega)$ of the coordinate [8,9]:

$$\overline{|x_{nk}|^2} \approx \frac{S_x(E, \omega)}{\hbar\rho(E)}. \tag{12}$$

The power spectrum $S_x(E, \omega)$ is related to the autocorrelation function of the coordinate for chaotic motion within a given component $C(E)$ on the energy surface $H = E$,

$$B_x(E, \tau) = \langle x(t + \tau) x(t) \rangle - \langle x(t) \rangle^2 \tag{13}$$

by the equation

$$S_x(E, \omega) = \frac{1}{2\pi} \int_{-\infty}^\infty B_x(E, \tau) e^{-i\omega\tau}\, d\tau. \tag{14}$$

The $\rho(E)$ in Eq. (12) denotes the density of states with the support in the same chaotic component $C(E)$ [8,10]. In the classical limit it is given by the expression

$$\rho(E) = \frac{1}{(2\pi\hbar)^2} \int_{C(E)} \delta(E - H(\vec{p}, \vec{r})) d\vec{p}\, d\vec{r}, \tag{15}$$

where $\delta(z)$ is the Dirac delta function, $H(\vec{p}, \vec{r})$ is the classical Hamiltonian function, and the integration is carried over the chaotic component $C(E)$. In the semiclassical case the matrix elements x_{nk} between the states with nonoverlapping supports vanish [10]. Therefore the same expression (15) for $\rho(E)$ should be inserted in the right-hand side (rhs) of Eq. (11).

For the energy E in Eq. (12) we take its interpolation value $\overline{E}_\pm = (E_n + E_{k\pm})/2 = E_n \pm (\hbar\omega)/2$ [8] (see Fig. 1). From Eqs. (10)–(12) in the limit $\hbar \to 0$ we obtain the expression for the rate of the energy absorption:

$$Q = \frac{\pi}{2} \omega^2 f_0^2 \left(\frac{\partial S_x(E, \omega)}{\partial E} + S_x(E, \omega) \frac{d \ln \rho(E)}{dE} \right). \tag{16}$$

By comparison of Eqs. (9) and (16), the following expression for the imaginary part of the susceptibility is obtained:

$$\alpha''(\omega) = \pi\omega \left(\frac{\partial S_x(E, \omega)}{\partial E} + S_x(E, \omega) \frac{d \ln \rho(E)}{dE} \right). \tag{17}$$

Let's introduce the notation

$$S_{2n}(E) = \int_{-\infty}^\infty \omega^{2n} S_x(E, \omega)\, d\omega \tag{18}$$

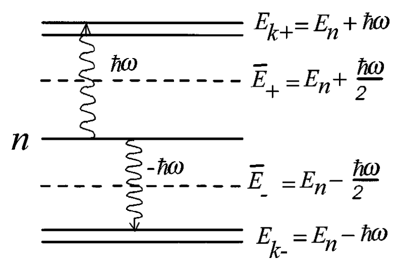

FIG. 1. Level scheme for the calculation of the rate of energy absorption. The interpolation values of energy $\overline{E}_\pm$ are shown by the dashed lines.

for even moments $S_{2n}(E)$ of the power spectrum of the coordinate. The substitution of Eq. (17) into Eqs. (7) and (8) yields

$$\gamma = -\frac{1}{m} + \frac{dS_2}{dE} + S_2 \frac{d\ln\rho(E)}{dE}, \tag{19}$$

$$\gamma\sigma^2 = \frac{1}{m^2}\langle U_{xx}\rangle - \frac{dS_4}{dE} - S_4\frac{d\ln\rho(E)}{dE}. \tag{20}$$

Formulae (17), (19), and (20) define the susceptibility of a chaotic system through the power spectrum of the coordinate, average value of the second derivative of the potential, and the logarithmic derivative of the density of states. Since the density of states in the semiclassical limit given by Eq. (15) is just proportional to $\hbar^{-2}$, its logarithmic derivative does not depend on $\hbar$ and hence is a purely classical quantity.

As an example of chaotic systems we take the Pullen-Edmonds model [11] with the Hamiltonian function

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{m\Omega^2}{2}(x^2 + y^2 + \lambda^{-2}x^2y^2). \tag{21}$$

Its chaotic properties have been thoroughly studied both numerically [12,13] and analytically [14,15]. In what follows we put $m = \Omega = \lambda = 1$.

For high energies $E \geq 10$, the motion of the system (21) is nearly ergodic: The invariant measure of the chaotic component at the Poincare section μ_s deviates from 1 by no more than 3×10^{-2} [12]. For this model even moments of the power spectrum of the coordinate have been found analytically in the ergodic approximation [15]. The substitution of the ergodic approximation for $S_2(E)$ into Eq. (19) yields $\gamma \equiv 0$. Since the Pullen-Edmonds system is only approximately ergodic, one can expect that the sum of the last two terms in the rhs of Eq. (8) deviates from its ergodic value ($= 1$) by a quantity of order $1 - \mu_s \sim 3 \times 10^{-2}$. An attempt to calculate γ from the numerically found values of $S_2(E)$ yielded the inconclusive value $\gamma = 0.07 \pm 0.47$. At present we cannot affirm that γ differs from zero.

The power spectrum of the coordinate has been arrived at directly from definition (14) by numerical integration of the canonical Hamilton equations. Six fragments of trajectory with lengths of 2×10^4 time units have been used to calculate $B_x(E, \tau)$ at each of two close energy values. The values of the density of states $\rho(E)$ have been taken from the ergodic approximation. The real part of the susceptibility has been calculated from the dispersion relation (6) with $\gamma = 0$. Figure 2 presents the real and imaginary parts of the susceptibility of the Pullen-Edmonds system at the energy $E = 16$. The spectral bands of negative absorption should be noted.

The presence of such bands may be typical for chaotic systems. As a second example we take the chaotic billiards [16,17]—systems in which the particle moves freely in two dimensions in a region confined by rigid walls. The power spectra of coordinates for chaotic billiards of nearly circular form closely resemble that of the Pullen-Edmonds model [18,19]. Since for the billiard $\rho(E) = \text{const}$, from Eq. (17) it follows that

$$\int_0^\infty \frac{\alpha''(\omega)}{\omega}\,d\omega = \frac{\pi}{2}\frac{dS_0}{dE} = \frac{\pi}{2}\frac{d\langle x^2\rangle}{dE} = 0. \tag{22}$$

This quality can hold only if there are bands with $\alpha''(\omega) < 0$. We note in passing that for chaotic billiards of any form, Eq. (19) yields $\gamma = 0$.

In conclusion we note that our constraint on the number of degrees of freedom (2) was chosen so as to ensure that the dynamic system has only one positive Lyapunov exponent. The results are valid also for Hamiltonian systems that describe the motion of the particle in a three-dimensional potential field if this condition is fulfilled.

We would like to thank E. A. Ostrovskaya for her useful discussions, and also V. P. Kandidov and A. S. Chirkin for their suggestions concerning the improvement of the numerical accuracy of our computer calculations. This work has been supported by the International Science Foundation (Grant No. ISF #M12300) and INTAS (Grant No. #94-2112).

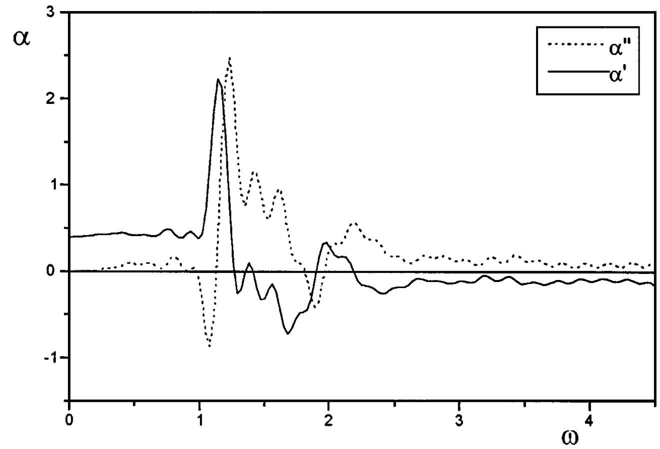

FIG. 2. The frequency dependence of the real (α', solid line) and imaginary (α'', dashed line) parts of the susceptibility of Pullen-Edmonds model (20) at the chaotic motion with energy $E = 16$.

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