Moduli, Scalar Charges, and the First Law of Black Hole Thermodynamics

Gary Gibbons*
DAMTP, Cambridge University, Silver Street, Cambridge CB3 9EW, United Kingdom

Renata Kallosh†
Department of Physics, Stanford University, Stanford, California 94305

Barak Kol‡
Department of Physics, Stanford University, Stanford, California 94305
(Received 24 July 1996)

We show that under variation of moduli fields ϕ the first law of black hole thermodynamics becomes $dM = \frac{\kappa dA}{8\pi} + \Omega dJ + \psi dq + \chi dp - \Sigma d\phi$, where Σ are the scalar charges. Also the Arnowitt-Desner-Misner mass is extremized at fixed A, J, (p,q) when the moduli fields take the fixed value $\phi_{\rm fix}(p,q)$ which depend only on electric and magnetic charges. Thus the double-extreme black hole minimizes the mass for fixed conserved charges. We can now explain the fact that extreme black holes fix the moduli fields at the horizon $\phi = \phi_{\rm fix}(p,q)$: $\phi_{\rm fix}$ is such that the scalar charges vanish: $\Sigma(\phi_{\rm fix},(p,q)) = 0$. [S0031-9007(96)01890-X]

PACS numbers: 04.70.Dy, 11.25.–w, 11.30.Pb

There has recently been intense interest in the thermodynamics of black holes in string theory. In particular the entropy S of some extreme black holes considered as a function of their conserved electric and magnetic charges (p,q) has been related to the logarithm of the number of the Bogomol'nyi-Prasad-Summerfeld (BPS) states at large (p,q) [1]. The properties of the black holes in the theories considered depend on the values ϕ_∞ of certain massless scalar fields, referred to as moduli fields, at spatial infinity. The moduli at infinity ϕ_∞ may be thought of as labeling different ground states or vacua of the theory. It is of crucial importance for the consistency of the state counting interpretation that the entropy $S = \frac{1}{4}A$, where A is the area of the event horizon, is independent (in the extreme limit) of the particular vacuum or ground state, i.e., of ϕ_∞, and depends only on the conserved charges (p,q). The Arnowitt-Desner-Misner (ADM) mass M, however, does depend on ϕ_∞ even in the extreme case. In the nonextreme case both the mass M and the area A depend in a nontrivial way on ϕ_∞. In other words, to specify completely a black hole in these theories one needs to give the entropy $S = \frac{1}{4}A$, the conserved charges (p,q), moduli at infinity ϕ_∞, and the total angular momentum J. In thermodynamic terms $A, (p^\Lambda, q_\Lambda), J, \phi^a_\infty$ are coordinates on the state space $\mathbf{R}_+ \times \mathbf{R}^{2n} \times \mathbf{R} \times \mathcal{M}_\phi$, where $\Lambda = 1, \dot{s}, n$ is the number of electric (or magnetic) charges, $\mathcal{M}_\phi$ is the manifold in which the scalars take their values, and $a = 1, \dot{s}, m = \dim \mathcal{M}_\phi$.

The usual first law of thermodynamics relates the variation of M to the temperature $T = \frac{\kappa}{2\pi}$, where κ is the surface gravity, the angular velocity Ω, and the electrostatic and magnetostatic potentials ψ^Λ and χ_Λ:

$$dM = \frac{\kappa dA}{8\pi} + \Omega dJ + \psi^\Lambda dq_\Lambda + \chi_\Lambda dp^\Lambda. \qquad (1)$$

However, Eq. (1) does not take into account the dependence upon the moduli ϕ_∞. It should clearly be replaced by

$$dM = \frac{\kappa dA}{8\pi} + \Omega dJ + \psi^\Lambda dq_\Lambda + \chi_\Lambda dp^\Lambda + \left(\frac{\partial M}{\partial \phi^a}\right) d\phi^a, \qquad (2)$$

where the partial derivative of the mass is taken at fixed values of the area, angular momentum and charges: $\left(\frac{\partial M}{\partial \phi^a}\right)_{A,J,p,q}$. Similar equations appeared in [2].

Our first result is that the coefficient of $d\phi^a$ is given by

$$\left(\frac{\partial M}{\partial \phi^a}\right)_{A,J,p,q} = -G_{ab}(\phi_\infty)\Sigma^b, \qquad (3)$$

where G_{ab} is the metric on the scalar manifold $\mathcal{M}_\phi$ in terms of which the kinetic part of the scalar Lagrangian density is

$$\frac{1}{2} G_{ab}\partial_\mu \phi^a \partial_\nu \phi^b g^{\mu\nu}\sqrt{-g}, \qquad (4)$$

and Σ^a are the scalar charges of the black hole defined by

$$\phi^a = \phi^a_\infty + \frac{\Sigma^a}{r} + O\left(\frac{1}{r^2}\right) \qquad (5)$$

at spatial infinity. Note that the scalar charges Σ^a themselves depend nontrivially on $A, (p^\Lambda, q_\Lambda), J, \phi^a_\infty$. The vector part of the Lagrangian is

$$-\frac{1}{4}(\mu_{\Lambda\Sigma}\mathcal{F}^\Lambda\mathcal{F}^\Sigma - \nu_{\Lambda\Sigma}\mathcal{F}^{\Lambda *}\mathcal{F}^\Sigma)\sqrt{-g}, \qquad (6)$$

where the Abelian field strengths are $\mathcal{F}^\Lambda \equiv \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$ and $*\mathcal{F}^\Sigma$ are the dual field strengths of the vector

 0031-9007/96/77(25)/4992(4)$10.00

fields and $\mu_{\Lambda\Sigma}$ and $\nu_{\Lambda\Sigma}$ are moduli dependent $n \times n$ matrices. The charges (q_Λ, p^Λ) are defined by

$$p^\Lambda = \frac{1}{4\pi}\int \mathcal{F}^\Lambda,$$
$$q_\Lambda = \frac{1}{4\pi}\int (\mu_{\Lambda\Sigma} * \mathcal{F}^\Sigma + \nu_{\Lambda\Sigma}\mathcal{F}^\Sigma). \quad (7)$$

We would like to stress that the charges must be defined as above, in order that Gauss's theorem holds; i.e., the charges are conserved and are the subject to quantization conditions in the quantum theory.

One may prove Eq. (2), with (3) and (4), either using Hamiltonian methods, modifying the procedure of Wald [3], or by covariant methods, following the older procedure of Bardeen, Carter, and Hawking [4]. A recent account of the covariant approach including scalars but dropping the last terms of Eq. (2) is given in [5]. From Eq. (95) of [5] for gravity coupled to a σ model we have

$$dM - \frac{\kappa dA}{8\pi} - \Omega dJ = -\oint d\phi^a G_{ab}(\phi)\frac{\partial\phi^b}{\partial x^i d\sigma^i}, \quad (8)$$

where the integral on the right hand side is over the boundary of a spacelike surface. The boundary has two components, one on the horizon and one at spatial infinity. The contribution from the horizon vanishes because ϕ^b is assumed to be independent of time and regular. The term at infinity yields

$$dM - \frac{\kappa dA}{8\pi} - \Omega dJ = -\Sigma^a G_{ab} d\phi^b. \quad (9)$$

If vectors are present there is the usual additional term due to variation of the charges.

The last term in Eq. (9) was dropped in [5] because in the application the authors had in mind (Skyrmion black hole) the scalar charges Σ^a do indeed vanish.

For black holes in string theory, however, the scalar charges Σ^a will not in general vanish. They will vanish if and only if ϕ_∞, and hence the vacuum state, is chosen to extremize the ADM mass at the fixed entropy $\frac{A}{4}$, angular momentum J, and conserved electric and magnetic charges (p^Λ, q_Λ). Note that despite the extra term in the first law the integrated version, i.e., the Smarr formula, remains [6]

$$M = \frac{\kappa A}{4\pi} + 2\Omega J + \psi^\Lambda q_\Lambda + \chi_\Lambda p^\Lambda. \quad (10)$$

From now on we will, for simplicity, consider only static nonrotating black holes. The extension to include rotation is both obvious and immediate.

The idea of extremization of the black hole mass in the moduli space at the fixed charges was suggested for supersymmetric black holes by Ferrara and one of the authors [7]. This idea is extended here for general black holes.

Our second result is that subject to a convexity condition that we explain below, the scalar charges vanish and hence M is extremal if and only if the black hole solution has constant values of the moduli fields

$$\phi^a(x) = \phi^a_\infty. \quad (11)$$

Moreover, the constant value ϕ^a_∞ is not arbitrary but must be chosen to extremize at fixed electric and magnetic charges a certain non-negative function V which is quadratic in the electric and magnetic charges and depends nontrivially on the scalars.

$$V = (p,q)^t \mathcal{M}\begin{pmatrix} p \\ q \end{pmatrix}, \quad (12)$$

where

$$\mathcal{M}^{-1} = \begin{vmatrix} \mu + \nu\mu^{-1}\nu & \nu\mu^{-1} \\ \mu^{-1}\nu & \mu^{-1} \end{vmatrix}. \quad (13)$$

For extended supergravity theories these $2n \times 2n$ moduli dependent matrices have been studied before [6–8].

Spherically symmetric nonextreme black holes in theories described above can be conveniently cast into the form [10]

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left[\frac{c^2 d\tau^2}{\sinh^4 c\tau} + \frac{c^2}{\sinh^2 c\tau}(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \quad (14)$$

The coordinate τ runs from $-\infty$ (horizon) to 0 (spatial infinity). The boundary condition for U is that $U(0) = 1$ and $U \to c\tau$ as $\tau \to -\infty$. The boundary condition for $\phi^a(\tau)$ is that $\phi^a(0) = \phi^a_\infty$ and $\frac{d\phi^a}{d\tau} = O(e^{c\tau})$ as $\tau \to -\infty$. The physical significance of c is that

$$c = \frac{\kappa A}{4\pi} = 2ST. \quad (15)$$

The field equations for U and ϕ^a are

$$\frac{d^2U}{d\tau^2} = 2V(\phi,(p,q))e^{2U}, \quad (16)$$

$$\frac{D\phi^a}{D\tau^2} = \frac{\partial V}{\partial\phi^a}e^{2U}, \quad (17)$$

and

$$\left(\frac{dU}{d\tau}\right)^2 + G_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} - V(\phi,(p,q))e^{2U} = c^2. \quad (18)$$

Our convexity condition is that the symmetric tensor field on $\mathcal{M}_\phi$ defined by

$$V_{ab} = \nabla_a\nabla_b V, \quad (19)$$

where ∇_a is the Levi-Civita covariant derivative with respect to the metric G_{ab} of $\mathcal{M}_\phi$, is non-negative.

It follows from the equation of motion for ϕ^a that

$$\frac{d^2V}{d\tau^2} = V_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} + 2e^{2U}\frac{\partial V}{\partial\phi^a}\frac{\partial V}{\partial\phi^b}G^{ab}. \quad (20)$$

If we multiply by V, integrate, and use the boundary conditions, we obtain

$$-\int_{-\infty}^{0}\frac{1}{2}\left(\frac{dV}{d\tau}\right)^2 d\tau = \Sigma^a\left(\frac{\partial V}{\partial\phi^a}\right)_\infty + \int_{-\infty}^{0}\left(V_{ab}\frac{d\phi^a}{d\tau}\frac{d\phi^b}{d\tau} + 2e^{2U}\frac{\partial V}{\partial\phi^a}\frac{\partial V}{\partial\phi^b}G^{ab}\right)d\tau. \quad (21)$$

If we assume that $\Sigma^a = 0$ and V_{ab} is positive definite, we must have $\frac{\partial V}{\partial\phi^a} = 0$ for all τ, which implies that the moduli are frozen, i.e., $\phi^a(r) = \phi^a_\infty$.

The mass of the black hole is given by $M = (\frac{dU}{d\tau})_{\tau=0}$, and therefore we have from (18) a rather useful general relation [10] which may be interpreted as the statement that the total self force on the hole due to the attractive forces of gravity and the scalar fields is not exceeded by the repulsive self force due to the vectors and vanishes only in the extreme case:

$$M^2 + G_{ab}\Sigma^a\Sigma^b - V(\phi^a_\infty) = 4S^2T^2. \quad (22)$$

One might refer to the inequality obtained from the nonnegativity of the right hand side as an antigravity bound. Note that unlike the Bogomol'nyi bound [9] its derivation requires neither supersymmetry nor duality invariance. Differentiating with respect to ϕ^a_∞ gives

$$M\frac{\partial M}{\partial\phi^c_\infty} + G_{ab}\Sigma^a\nabla_c\Sigma^b - \frac{1}{2}\frac{\partial V}{\partial\phi^c_\infty} = 4S^2T\frac{\partial T}{\partial\phi^c_\infty}. \quad (23)$$

We deduce that if the mass is extremized with respect to ϕ^a_∞, then so is the temperature. It follows that the fixed or "frozen" moduli must minimize V, i.e., $\phi_{\rm fix}$ is defined by

$$\left(\frac{\partial V}{\partial\phi^a}\right)_{\phi=\phi_{\rm fix},(p,q)} = 0. \quad (24)$$

Static black holes with frozen moduli have the spacetime geometry given by the Reissner-Nordström metric.

A year ago Ferrara, Kallosh, and Strominger [11] found that for a class of supersymmetric black holes the moduli field at the horizon ϕ_H depends only on the conserved electric and magnetic charges

$$\phi_{H,\rm extreme} = \phi_{\rm fix}(p,q). \quad (25)$$

Recall that at extremality, the mass depends on the moduli at infinity and the conserved charges

$$M = M_{\rm extreme}[\phi^a_\infty,(p,q)]. \quad (26)$$

An implicit formula was found more recently in [7] for $\phi_{H,\rm extreme}$ that can be written as

$$\left(\frac{\partial M_{\rm extreme}}{\partial\phi}\right)_{(p,q),\phi=\phi_{H,\rm extreme}} = 0, \quad (27)$$

where the derivative is taken at fixed values of charges. The result which holds for all of the theories we consider here was found by analyzing the radial equation for the moduli fields $\phi(r)$ which is governed by the function $V(\phi,p,q)$.

Two questions arose and motivated the results of this paper:

(i) Why is $\phi_{H,\rm extreme}$ independent of ϕ_∞?

(ii) Why is $\phi_{H,\rm extreme}$ given by (27)?

We can now offer an answer for the second question. From (3), which we first derived for the example given in [12], it follows that Eq. (27) is equivalent to

$$\Sigma^a(\phi_{\rm fix},(p,q)) = 0, \quad (28)$$

thus defining $\phi_{\rm fix} = \phi_{\rm fix}(p,q)$. (This equation was noted in [7] for the extreme case.) But as we stated above, a black hole with vanishing scalar charge must have spatially constant moduli fields: $\phi^a(r) = \phi^a_{H,\rm extreme} = \phi^a_\infty$, or frozen moduli. In other words, to satisfy Eq. (28) we must choose ϕ^a_∞ to be $\phi^a_{H,\rm extreme}$.

As found in [7], the entropy of all extreme black holes is independent of ϕ^a_∞ and is given by

$$S = \frac{A}{4} = \pi V(\phi_{\rm fix}(p,q),(p,q)). \quad (29)$$

Our new result establishes that for any static black hole, extreme or not,

$$M(S,\phi_\infty,(p,q)) \geq M(S,\phi_{\rm fix},(p,q)). \quad (30)$$

But because black holes with frozen moduli have the Reissner-Nordström geometry, the right hand side of (16) is always greater than the mass of the extreme Reissner-Nordström black hole with same charges.

We would like to emphasize that our results hold for a wide class of theories—one need not assume either supersymmetry or duality invariance. In addition we wish to emphasize the following:

(i) The scalar charges Σ^a are not conserved but they do act as the sources for the moduli. They are not associated with a conserved current. The flux of the gradient of the scalar charge vanishes at the horizon. Thus the scalar charge resides entirely outside the event horizon.

(ii) Previously one did not consider variations of the moduli at infinity ϕ_∞, which were regarded fixed once and for all. In that case the scalar charge Σ^a need not be specified independently of the mass, angular momentum, and electric and magnetic charges. However, if one does not regard the moduli at infinity to be given *a priori* one needs to specify, in addition to M, J, and (q,p), either

ϕ_∞ or Σ^a to characterize completely the black hole. This may be important when considering situations in which ϕ_∞ becomes dynamical, for example, if one considers slow adiabatic changes of ϕ_∞ or possibly time-dependent cosmological situations.

Stimulating discussions with S. Ferrara, A. Linde, A. Peet, A. Rajaraman, L. Susskind, and W.K. Wong are gratefully acknowledged. B.K. thanks his teacher L. Susskind, Arvind Rajaraman, and Edi Halyo. G.W.G. would like to thank the Stanford Theoretical Physics Institute for the hospitality and support during the period of this work. This work is supported by the NSF Grant No. PHY-9219345.

*Electronic address: G.W.Gibbons@damtp.cam.ac.uk
†Electronic address: kallosh@renata.stanford.edu
‡Electronic address: barak@leland.stanford.edu

[1] A. Strominger and C. Vafa, Report No. hep-th/9601029.
[2] J.D.E. Creighton and R.B. Mann, Phys. Rev. D **52**, 4569 (1995); Report No. gr-qc/9505007; Report No. gr-qc/9511012.
[3] R.M. Wald, Report No. gr-qc/9305022.
[4] J.H. Bardeen, B. Carter, and S.W. Hawking, Commun. Math. Phys. **31**, 181 (1973).
[5] M. Heusler and N. Straumann, Class. Quantum Gravity **10**, 1299 (1993); Phys. Lett. **315**, 55 (1993).
[6] P. Breitenlohner, D. Maison, and G. Gibbons, Commun. Math. Phys. **120**, 295 (1988).
[7] S. Ferrara and R. Kallosh, Report No. hep-th/9602136; Report No. hep-th/9603090.
[8] R. Kallosh, M. Shmakova, and W.K. Wong, Report No. hep-th/9607077.
[9] G. Gibbons and C.M. Hull, Phys. Lett. **109B**, 190 (1982).
[10] G. Gibbons, Nucl. Phys. **B207**, 337 (1982).
[11] S. Ferrara, R. Kallosh, and A. Strominger, Phys. Rev. D **52**, 5412 (1995); Report No. hep-th/9508072.
[12] R. Kallosh, A. Linde, T. Ortin, A. Peet, and A. Van Proeyen, Phys. Rev. D **46**, 5278 (1992).