# Early Crisis Induced in Maps with Parametric Noise

H. L. Yang and Z. Q. Huang
*Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China*

E. J. Ding
*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, China*
*Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing 100875, China*
*Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China*
(Received 12 August 1996)

In this paper a new type of crisis in random maps is studied. The trigger of this new crisis is the tunnel effect induced by a backward tangent bifurcation. This is different from the formerly reported crises caused by the collision of the chaotic attractor with an unstable orbit. The reasons why the characteristic time of this new crisis is super long are given. Another case of crisis triggered by the random collision of the attractor with the system's trapping region boundary can also be found in this model. The two cases of crisis can transform into each other by continuously varying the control parameter. [S0031-9007(96)01842-X]

PACS numbers: 05.45.+b

For a dynamical system, one of the most fundamental problems is to study the change in its behavior as some parameters of the system are varied. The sudden qualitative change in the chaotic dynamical behavior induced by variations of the parameters, named a crisis, has been found to be common in dynamical systems, and many works have been done on this subject [1–6]. Three prototypes of crises are given by Grebogi *et al.* [2–6]. The first one is the sudden destruction of a chaotic attractor, caused by the collision with an unstable orbit at the boundary of this attractor. It was called a boundary crisis. The second one is the sudden increase of the size of a chaotic attractor in the phase space, named an interior crisis. It occurs when the chaotic attractor collides with a periodic orbit in the interior of its basin. The third one is the merging of more than one chaotic attractor, named a merging crisis. It is caused by the simultaneous collision of the attractors with the boundary which separates them. For a properly defined characteristic time, it exhibits a perfect power law relation with the variation of the control parameter. A quantitative theory for the determination of the critical exponents is developed in Ref. [2]. For a system under the influence of a random noise, a crisis can even appear when the control parameter does not reach the critical value [4,5]. This is the noise induced crisis.

It should be noted that for all these types of crises, it is the collision of the chaotic attractor with an unstable orbit, or equivalently, its stable manifold, that triggers the crisis. In this paper, a new type of crisis in a random system caused by the backward tangent bifurcation is reported. For this case, the phase points are bounded in separated bands of the attractor before the occurrence of crisis. With the increasing of the control parameter, a stable fixed point (node) on the attractor and an unstable fixed point (saddle) on the tapping region boundary will come closer and closer together. And they will annihilate finally due to a tangent (saddle-node) bifurcation. As a result, a narrow tunnel will appear occasionally between the map function $y = F(y, z_n, z_{n+1})$ and the $45°$ line. Then, the phase points in one band will be able to transfer to other bands through this randomly appearing narrow tunnel. Since the tunnel is induced by a backward tangent bifurcation, we call this new crisis a tunnel crisis induced by a backward tangent bifurcation. Here, the tunnel effect, instead of the collision of the attractor with an unstable orbit in former cases, is the trigger of crisis. We will use a simple example below to illustrate the cause and properties of this new crisis.

The model we studied is just a random map [7–15],

$$y_{n+1} = f(y_n, z_n), \tag{1}$$

where $z_n$ is a time dependent random number. One of our motivations to study such a system driven by a random noise is to understand the mechanism of the synchronization in chaotically driven systems [11]. For the chaotic synchronization, there is always an ensemble of identical nonlinear systems driven by a chaotic signal. So, the study of a single random system will be important to the understanding of its mechanism. This work is also motivated by the attempt to study the domain loss of a multiperiodic system under the influence of a random driving [7–10]. Some authors have even noted that it is a crisislike phenomenon [7–9].

In this paper, we just study the simplest logistic map driven by a random noise,

$$f(y_n, z_n) = z_n y_n (1 - y_n), \tag{2}$$

$$z_n = b + a x_n, \tag{3}$$

where $a$, $b$ are two positive real constants, and $x_n$ is a random number homogeneously distributed in the interval

  

$[0, 1]$. To keep the value of $y_n$ bounded in the interval $[0, 1]$, the inequalities

$$a > 0, \qquad b > 0, \qquad a + b < 4 \tag{4}$$

should be satisfied. These conditions define a triangle in the $(a, b)$ plane (cf. Fig. 1). Two curves formed by zero points of the Lyapunov exponent divide the triangle into three regions. As reported before, just beyond these two curves, two different manifestations of on-off intermittency can be found [12–15]. For $a, b$ lying in the region between the two curves, an ensemble of different initial points will clap to a single one after a long period of transient iterations. On the other hand, just on the $b$ axis, there is a cascade of period doubling bifurcations as well as many period windows. They disappear completely for the strong influence of random driving. What will be the crossover of these two situations? The behavior of a period-2 system under the influence of a random driving with increasing strength is shown below.

For $a = 0; 3.0 < b < 3.449\ldots$, the system we studied is just of period two. If we fix $b = 3.2$, and slowly increase $a$ from zero, the two fixed points will be slowly blurred into two bands of continuously changed sizes. With further increased $a$, the two bands will suddenly expand their sizes and merge into one when $a$ passes through a critical value $a_3^* = 0.175$. The bifurcation diagram is shown in Fig. 2. Since the parameters of interest for us are limited in the period 2 interval, we shall only consider the double iteration of the original map as below,

$$y = F(y, z_n, z_{n+1}) \equiv f(f(y, z_n), z_{n+1}). \tag{5}$$

For random variations of $z_n$ and $z_{n+1}$, the maps $F(y, z_n, z_{n+1})$ at different steps are blurred into a band bounded by $F(y, b, b), F(y, b, a + b), F(y, a + b, b)$, and $F(y, a + b, a + b)$. So the two fixed points of the deterministic map are blurred into two bands correspondingly. The boundaries of the two bands can be calculated rigorously [16]. For example, the boundaries of the lower band are

$$B_1 = \begin{cases} y_{b,a+b,1}, & \text{if } 0 \le a < a_1^*, \\ y_{b,a+b,1}, & \text{if } a_1^* \le a < a_2^*, \\ F(B_2, b, a + b), & \text{if } a_2^* \le a < a_3^*, \end{cases} \tag{6}$$

$$B_2 = \begin{cases} y_{a+b,b,1}, & \text{if } 0 \le a < a_1^*, \\ F(0.5, a + b, b), & \text{if } a_1^* \le a < a_2^*, \\ F(0.5, a + b, b), & \text{if } a_2^* \le a < a_3^*, \end{cases} \tag{7}$$

where $B_1$ (or $B_2$) is the upper (or lower) boundary of this band, and $y_{b,a+b,i}$ is the $i$th root of the equation $F(y, b, a + b) - y = 0$ when all its nonzero roots are arranged in ascending order. Other symbols of $y$ with subscripts of similar meaning will be used in this paper. The critical values of $a$ satisfy

$$F'(y_{a_1^*+b,b,1}, a_1^* + b, b) = 0,$$
$$F(B_2, b, a_2^* + b) = y_{b,a_2^*+b,1}, \tag{8}$$
$$y_{b,a_3^*+b,1} = y_{b,a_3^*+b,2},$$

where $F'(y, a + b, b) \equiv \partial_y F(y, a + b, b)$. For $b = 3.2$, the three critical values of $a$ are, respectively, $a_1^* = 0.024$, $a_2^* = 0.092$, $a_3^* = 0.175$. While for $b = 3.1$, the three critical values degenerate into a single one $a_1^* = a_2^* = a_3^* = 0.051$. For values of $a$ smaller than $a_3^*$, the system has two trapping regions distributed separately in two intervals $[F(0.5, a + b, b), y_{b,a+b,2}]$ and $[y_{a+b,b,2}, f(0.5, a + b)]$ [cf. Fig. 3(a)]. All points iterated into these regions will stay there forever. So the attractor of the system can only be two separated bands. With increasing values of $a$, the stable fixed point $y_{b,a+b,1}$ on the attractor and the unstable fixed point $y_{b,a+b,2}$ on the trapping region boundary will come closer

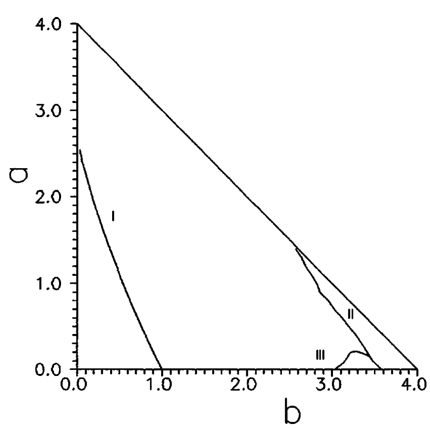


FIG. 1. The phase diagram of the random logistic map. The curves $I$ and $II$ are formed by zero points of the Lyapunov exponent. The curve $III$ is the critical curve of a new type of crisis.

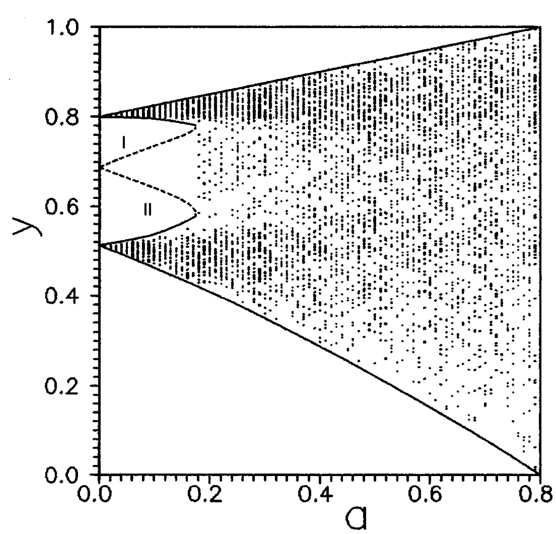


FIG. 2. The bifurcation diagram of the random map with $b = 3.2$. The solid curves on the boundary are calculated from Eqs. (6)–(8). The dashed lines $I$ and $II$ are formed by unstable fixed points $y_{b,a+b,2}$ and $y_{a+b,b,2}$, respectively. They annihilate at the crisis point due to a backward tangent bifurcation.

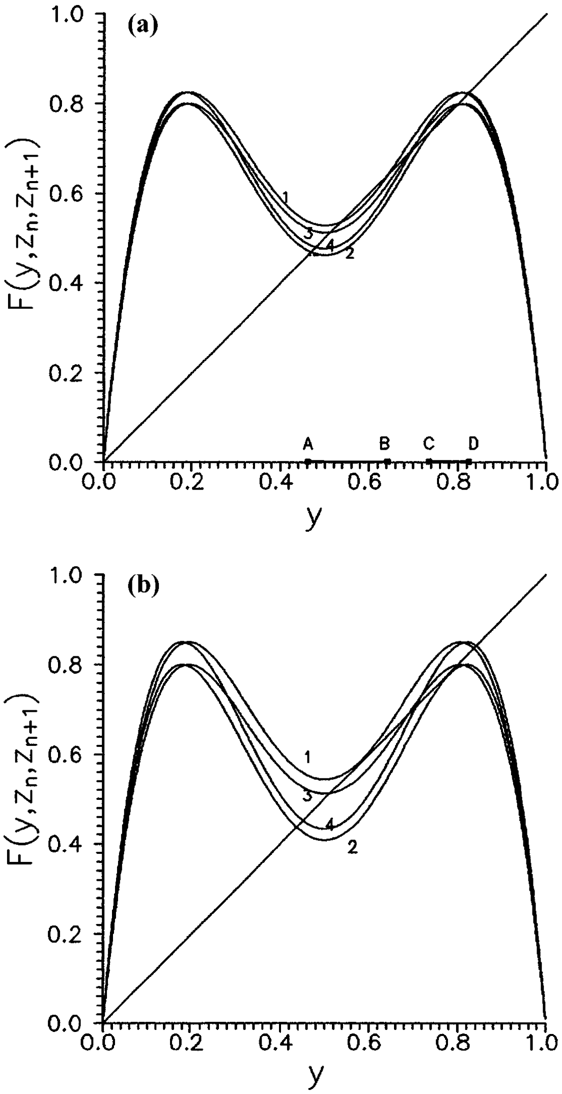


FIG. 3. The map functions (1) $y = F(y, b, a + b)$, (2) $y = F(y, a + b, b)$, (3) $y = F(y, a + b, a + b)$, (4) $y = F(y, b, b)$, with $b = 3.2$ and (a) $a = 0.1$; (b) $a = 0.2$. The four points A $[F(0.5, a + b, b)]$, B $(y_{b,a+b,2})$, C $(y_{a+b,b,2})$, D $[f(0.5, a + b)]$ in (a) are the boundary points of trapping regions. It can be seen in (b) that a narrow tunnel appears between $y = F(y, b, a + b)$ and the 45° line.

and closer together. And they will finally annihilate due to a backward tangent bifurcation at the critical value $a_3^*$ (cf. Fig. 2). After that, a narrow tunnel will appear between the map function $y = F(y, b, a + b)$ and the 45° line [cf. Fig. 3(b)]. Then the points in one band can be transformed into another band through this narrow tunnel. Here, the "tunnel effect" caused by a backward tangent bifurcation is the trigger of this new crisis. It is different from the formerly reported cases which are caused by the collision of a chaotic attractor with an unstable orbit. This shows that the crisis we find here is a new one.

It should be noticed that the narrow tunnel between the iteration curve $y = F(y, z_n, z_{n+1})$ and the 45° line can only appear when some conditions are satisfied. For example, the conditions are $z_n \approx b$ and $z_{n+1} \approx b + a_3^*$ for one slightly greater than $a_3^*$. For random variations of $z_n$, the tunnel can only appear with a small probability.

Similar to the former studies of the crisis behavior, we can define the characteristic time $\tau$ as the number of steps of the iterations which an orbit spends in one band before it tunnels into another band. The numerical values of $\tau$ calculated with the various values of $\epsilon = a - a_3^*$ are shown in Fig. 4. It can be seen that the mean time $\langle\tau\rangle$ increases very quickly with the decrease of $\epsilon$. It is about $10^7$ for $\epsilon \sim 0.1$ with $b = 3.1$. A numerical fitting of data shows that $\ln\langle\tau\rangle \sim 1/\epsilon$; i.e., we have $\langle\tau\rangle \sim \exp(c/\epsilon)$. This relation can be deduced rigorously from a stochastic bistable model [16]. The scaling relation of $\langle\tau\rangle$ with $\epsilon$ shows that the crisis reported here is different from the "unstable-unstable pair bifurcation crisis" given by Grebogi *et al.* That crisis has the characteristic scaling of $\langle\tau\rangle$ with $\epsilon$ given by $\langle\tau\rangle \sim \exp(k/\epsilon^{1/2})$ [17].

The origin of this super long characteristic time may be as follows: First, for random variations of $a$ and $b$, the tunnel between the map function $y = F(y, b, a + b)$ and the 45° line can appear only occasionally. Second, for values of $a$ slightly greater than $a_3^*$, it takes a long period of iterations for phase points to trasverse the narrow tunnel. During this long period of iterations, the tunnel can be maintained only with a very small probability.

With the variation of $b$, the critical points $a_3^*$ will form a curve in the $(a, b)$ plane (cf. Fig. 1). This curve is

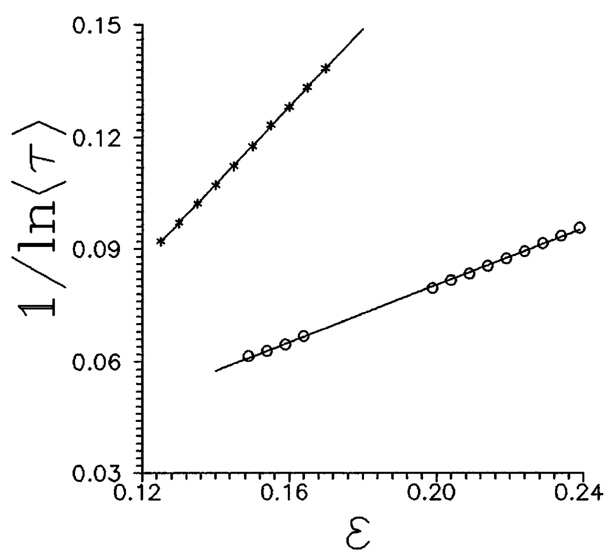


FIG. 4. The mean characteristic time $\langle\tau\rangle$ during which the system stays in one band vs $\epsilon = a - a_3^*$. The two curves are of $b = 3.2$ (*) and $b = 3.1$ (∘), respectively. Each point in this figure is the average of 10 000 examples. The unit of $\tau$ is the step of iteration.

just the critical curve of the new crisis reported here. For parameters below this curve, the orbit iterates in two separated bands. While for parameters passing through this curve, the two separated bands will suddenly expand their sizes and merge into one. For an ensemble of phase points of different initial values, the merging of two bands means that all of the orbits will be synchronous after the transient iterations. So, this critical curve of crisis is also the critical curve of synchronization. This implies that the new crisis may be one of the mechanisms of synchronization (or domain loss) for multiperiodic systems under the influence of a random noise.

For $b = 3.3$, with increasing $a$, another type of crisis can be found. It is caused by the collision of the attractor with the boundary of the system's trapping region. The critical value of $a$ for this crisis is determined by the relation

$$F(F(0.5, a + b, b), b, a + b) = y_{b,a+b,2}\,. \qquad (9)$$

The numerical calculation of the characteristic time shows that it is also super long. By slowly decreasing $b$, this case of crisis can be transferred into the case caused by a "tunnel effect." A detailed study of this transition will be reported elsewhere [16].

It can be seen that under the influence of noise, the merging crisis of a deterministic map now occurs at an earlier time. For the deterministic map $f(y) = zy(1 - y)$, the merging crisis occurs at $z = 3.678\ldots$ when the two bands of the attractor collide with the unstable orbit $y^* = 1 - 1/z$ which separates them. While for the random map studied here, we have the critical value for merging crisis: $b + a_3^* = 3.2 + 0.175 = 3.375$. Under the influence of parametric noise, the control parameters at two consecutive steps of iterations become nonidentical. The tangent bifurcation, which triggers the new crisis, is just induced by this difference in the control parameters of two consecutive steps of iterations.

In conclusion, in this paper, a new case of crisis is found in a random map when we increase the strength of the random driving. It is triggered by the tunnel effect due to a backward tangent bifurcation. This is different from the mechanism of the crises reported formerly. Another crisis caused by the random collision of the attractor with its trapping region boundary can also be found in this model. The two cases can transform into each other with continuous variations of the parameter $b$.

This project is supported by the National Nature Science Foundation, the National Basic Research Project "Nonlinear Science," and the Education Committee of the State Council through the Foundation of Doctoral Training.

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