Universal Formula for Noncommutative Geometry Actions: Unification of Gravity and the Standard Model

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A universal formula for an action associated with a noncommutative geometry, defined by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, is proposed. It is based on the spectrum of the Dirac operator and is a geometric invariant. The new symmetry principle is the automorphism of the algebra $\mathcal{A}$ which combines both diffeomorphisms and internal symmetries. Applying this to the geometry defined by the spectrum of the standard model gives an action that unifies gravity with the standard model at a very high energy scale. [S0031-9007(96)01860-1]

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Riemannian geometry has played an important role in our understanding of space-time, especially in the development of the general theory of relativity. The basic data of Riemannian geometry consists of a *manifold M* whose points $x \in M$ are locally labeled by finitely many coordinates $x^\mu \in \mathbb{R}$, and in the infinitesimal *line element*, $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$. The dynamics of the metric is governed by the Einstein action, and the symmetry principle is diffeomorphism invariance. The other basic interactions consisting of strong, weak, and electromagnetic forces are well described by the standard model action, and the symmetry principle is the gauged internal symmetry. Therefore the natural group of invariance that governs the sum of the Einstein and standard model actions is the semidirect product of the group of local gauge transformations, $\mathcal{U} = C^\infty(M, \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3))$ by the natural action of Diff(M). Another concept which is vital to the construction of the standard model is spontaneous symmetry breaking and Higgs fields, but this has no geometrical significance.

In a new development in mathematics, it has been shown that Riemannian geometry could be replaced with a more general formulation, noncummutative geometry. The basic data of noncommutative geometry consists of an involutive *algebra* $\mathcal{A}$ of operators in Hilbert space $\mathcal{H}$ and of a self-adjoint unbounded operator D in $\mathcal{H}$ [1–6]. The geodesic distance between points is recovered by

$$d(x,y) = \mathrm{Sup}\{|a(x) - a(y)|; \quad a \in \mathcal{A}, \|[D,a]\| \le 1\},$$

which implies that the inverse D^{-1} of D plays the role of the infinitesimal unit of length *ds* of ordinary geometry.

There is no information lost in trading the original Riemannian manifold M for the corresponding spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M, $\mathcal{H} = L^2(M,S)$ the Hilbert space of L^2-spinors, and D the Dirac operator of the Levi-Civita spin connection. More importantly, it was shown in [4] that the axioms of commutative geometry characterizing spectral triples $(\mathcal{A}, \mathcal{H}, D)$ coming from the above spinorial construction could be given in a form generalizable to the noncommutative case which involves the dimension n of M. The parity of n implies a $\mathbb{Z}/2$ grading γ of the Hilbert space $\mathcal{H}$ such that

$$\gamma = \gamma^*, \quad \gamma^2 = 1, \quad \gamma a = a\gamma \quad \forall\, a \in \mathcal{A}, \quad \gamma D = -D\gamma .$$

Moreover, one keeps track of the *real structure* on $\mathcal{H}$ as an antilinear isometry J in $\mathcal{H}$ satisfying simple relations

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad J\gamma = \varepsilon''\gamma J; \quad \varepsilon, \varepsilon', \varepsilon'' \in \{-1, 1\},$$

where the values of $\varepsilon, \varepsilon', \varepsilon''$ are determined by n modulo 8. A great advantage in adopting noncommutative geometry is that it allows the study of spaces which could not be handled otherwise. The usual emphasis on the points $x \in M$ of a geometric space is now replaced by the spectrum $\Sigma \subset \mathbb{R}$ of the operator D. Indeed, if one forgets about the algebra $\mathcal{A}$ in the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ but retains only the operators D, γ, and J acting in $\mathcal{H}$ one can characterize this data by the spectrum Σ of D which is a discrete subset with multiplicity of $\mathbb{R}$. In the even case $\Sigma = -\Sigma$. The existence of Riemannian manifolds which are isospectral (i.e., have the same Σ) but not isometric shows that the following hypothesis is stronger than the usual diffeomorphism invariance of the action of general relativity: "The physical action depends only on Σ". The most natural candidate for an action that depends only on the spectrum is

$$\mathrm{Tr}\chi\left(\frac{D}{\Lambda}\right) + \langle \psi, D\psi \rangle, \tag{1}$$

where χ is a positive function and Λ is an arbitrary scale. This form of the action is dictated by the condition that it is additive for disjoint union. In the usual Riemannian case the group Diff(M) of diffeomorphisms of M is canonically isomorphic to the group Aut($\mathcal{A}$) of

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automorphisms of the algebra $\mathcal{A} = C^\infty(M)$. To each $\varphi \in \mathrm{Diff}(M)$ one associates the algebra preserving map $\alpha_\varphi : \mathcal{A} \to \mathcal{A}$ given by

$$\alpha_\varphi(f) = f \circ \varphi^{-1} \quad \forall\, f \in C^\infty(M) = \mathcal{A}\,.$$

In general the group $\mathrm{Aut}(\mathcal{A})$ of automorphisms of the involutive algebra $\mathcal{A}$ plays the role of the diffeomorphisms of the noncommutative (or spectral for short) geometry $(\mathcal{A}, \mathcal{H}, D)$. The first interesting new feature of the general case is that the group $\mathrm{Aut}(\mathcal{A})$ has a natural normal subgroup, $\mathrm{Int}(\mathcal{A}) \subset \mathrm{Aut}(\mathcal{A})$ (*which is analogous to the normal subgroup* $\mathcal{U}$ *of the symmetry group* $G = \mathcal{U} \rtimes \mathrm{Diff}(M)$), where an automorphism α is *inner* iff there exists a unitary operator $u \in \mathcal{A}$ $(uu^* = u^*u = 1)$, such that $\alpha(a) = uau^*\ \forall\, a \in \mathcal{A}$. This leads us to the postulate that *The symmetry principle in noncommutative geometry is invariance under the group Aut*$(\mathcal{A})$.

We now apply these ideas to derive a noncommutative geometric action unifying gravity with the standard model. The starting point is the spectrum of the standard model. The symmetries of the spectrum indicate that the algebra to be taken is $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ where the algebra $\mathcal{A}_F$ is *finite dimensional,* $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, and $\mathbb{H} \subset M_2(\mathbb{C})$ is the algebra of quaternions, $\mathbb{H} = \left\{\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}; \alpha, \beta \in \mathbb{C}\right\}$. $\mathcal{A}$ is a tensor product which geometrically corresponds to a product space; an instance of spectral geometry for $\mathcal{A}$ is given by the product rule

$$\mathcal{H} = L^2(M,S) \otimes \mathcal{H}_F, D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F\,, \quad (2)$$

where $(\mathcal{H}_F, D_F)$ is a spectral geometry on $\mathcal{A}_F$, while both $L^2(M,S)$ and the Dirac operator $\partial\!\!\!/_M$ on M are as above. The choice of $\mathcal{A}_F$ could be related to the quantum group $SU(2)_q$ at the cubic root of unity [4]. The Dirac operator $D = \partial\!\!\!/_M$ on M is taken, for simplicity, to correspond to a space without torsion.

The group $\mathrm{Aut}(\mathcal{A})$ of diffeomorphisms falls in equivalence classes under the normal subgroup $\mathrm{Int}(\mathcal{A})$ of inner automorphisms. In the same way the space of metrics has a natural foliation into equivalence classes. The *internal fluctuations* of a given metric are given by the formula

$$D = D_0 + A + JAJ^{-1}, \quad A = \Sigma a_i[D_0, b_i],$$
$$a_i, b_i \in \mathcal{A} \quad \text{and} \quad A = A^*. \quad (3)$$

Thus starting from $(\mathcal{A}, \mathcal{H}, D_0)$ with obvious notations, one leaves the representation of $\mathcal{A}$ in $\mathcal{H}$ untouched and just perturbs the operator D_0 by (3) where A is an arbitrary self-adjoint operator in $\mathcal{H}$ of the form $A = \Sigma a_i[D_0, b_i]$; $a_i, b_i \in \mathcal{A}$. For Riemannian geometry these fluctuations are trivial.

The hypothesis which we shall test in this Letter is that there exists an energy scale Λ in the range 10^{15}–10^{19} Gev at which we have a geometric action given by (1).

The quarks Q and leptons L are

$$Q = \begin{pmatrix} u_L \\ d_L \\ d_R \\ u_R \end{pmatrix}, \quad L = \begin{pmatrix} \nu_L \\ e_L \\ e_R \end{pmatrix}.$$

We now describe the internal geometry. The choice of the Dirac operator and the action of $\mathcal{A}_F$ in $\mathcal{H}_F$ comes from the restrictions that these must satisfy

$$J^2 = 1, \quad [J, D] = 0, \quad [a, Jb^*J^{-1}] = 0,$$
$$[[D, a],\ Jb^*J^{-1}] = 0 \quad \forall\, a, b\,. (4)$$

We can now compute the inner fluctuations of the metric and thus operators of the form $A = \Sigma a_i[D, b_i]$. This with the self-adjointness condition $A = A^*$ gives a U(1), SU(2), and U(3) gauge fields as well as a Higgs field. In the computation of $A + JAJ^{-1}$ one removes a U(1) part from the above gauge fields. One drops the trace part which does not affect the metric (see [4] for details). The Dirac operator D_q that takes the inner fluctuations into account is given by the 36×36 matrix (acting on the 36 quarks) (tensored with Clifford algebras)

$$D_q = \begin{bmatrix} \gamma^\mu \otimes (D_\mu \otimes 1_2 - \frac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha - \frac{i}{6} g_{01} B_\mu \otimes 1_2) \otimes 1_3, & \gamma_5 \otimes k_0^d \otimes H, & \gamma_5 \otimes k_0^u \otimes \tilde H \\ \gamma_5 \otimes k_0^{d^*} \otimes H^*, & \gamma^\mu \otimes (D_\mu + \frac{i}{3} g_{01} B_\mu) \otimes 1_3, & 0 \\ \gamma_5 k_0^{u^*} \tilde H^*, & 0, & \gamma^\mu \otimes (D_\mu - \frac{2i}{3} g_{01} B_\mu) \otimes 1_3 \end{bmatrix} \otimes 1_3$$
$$+ \gamma^\mu \otimes 1_4 \otimes 1_3 \otimes \left(-\frac{i}{2} g_{03} V_\mu^i \lambda^i\right), \quad (5)$$

where σ^α are Pauli matrices and λ^i are Gell-Mann matrices satisfying $\mathrm{Tr}(\lambda^i \lambda^j) = 2\delta^{ij}$. The vector fields B_μ, A_μ^α, and V_μ^i are the U(1), $SU(2)_w$, and $SU(3)_c$ gauge fields with gauge couplings g_{01}, g_{02}, and g_{03}. The differential operator D_μ is given by $D_\mu = \partial_\mu + \omega_\mu$ where $\omega_\mu = \frac{1}{4}\,\omega_\mu^{ab}\gamma_{ab}$ and $\gamma^\mu = e_a^\mu \gamma^a$. The Dirac operator of the Riemannian manifold M is taken to be that of a torsion free space; the more general case with torsion will not be considered here. The scalar field H is the Higgs doublet, and $\tilde H = (i\sigma^2 H)$ is the SU(2) conjugate of H.

The Dirac operator acting on the leptons, taking inner fluctuations into account, is given by the 9×9 matrix (tensored with Clifford algebra matrices),

$$D_\ell = \begin{bmatrix} \gamma^\mu \otimes (D_\mu - \frac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha + \frac{i}{2} g_{01} B_\mu \otimes 1_2) \otimes 1_3 & \gamma_5 \otimes k_0^e \otimes H \\ \gamma_5 \otimes k_0^{*e} \otimes H^* & \gamma^\mu \otimes (D_\mu + i g_{01} B_\mu) \otimes 1_3 \end{bmatrix}. \tag{6}$$

The matrices k_0^d, k_0^u, and k_0^e are 3×3 family mixing matrices. According to our universal formula the spectral action for the standard model is given by

$$\mathrm{Tr}[\chi(D^2/m_0^2)] + (\psi, D\psi), \tag{7}$$

where m_0 is a cutoff mass scale and $(\psi, D\psi)$ will include both the quark and leptonic interactions. It is a simple exercise to compute the square of the Dirac operator given by (5) and (6). This can be cast into the elliptic operator form [7]

$$P = D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \cdot \mathbb{I} + \mathbb{A}^\mu \partial_\mu + \mathbf{B}),$$

where $\mathbb{I}$, $\mathbb{A}^\mu$ and $\mathbf{B}$ are matrices of the same dimensions as D. Using the heat kernel expansion for

$$\mathrm{Tr}\, e^{-tP} \simeq \sum_{n \geq 0} t^{(n-m/d)} \int_M a_n(x, P)\, dv(x),$$

where m is the dimension of the manifold in $C^\infty(M)$, d is the order of P (in our case $m = 4$, $d = 2$), and $dv(x) = \sqrt{g}\, d^m x$ where $g^{\mu\nu}$ is the metric on M appearing in P, we can show that

$$\mathrm{Tr}\chi(P) \simeq \sum_{n \geq 0} f_n a_n(P),$$

where the coefficients f_n are given by

$$f_0 = \int_0^\infty \chi(u) u\, du, \quad f_2 = \int_0^\infty \chi(u)\, du,$$

$$f_{2(n+2)} = (-1)^n \chi^{(n)}(0), \quad n \geq 0,$$

and $a_n(P) = \int a_n(x, P)\, dv(x)$ are the Seeley–de Witt coefficients. $a_n(P)$ vanish for odd values of n and the first four a_n for even n are given in Gilkey [7]. A very lengthy but straightforward calculation, the details of which will be reported somewhere else [8] gives for the bosonic action

$$I_b = \int d^4x \sqrt{g} \Big[\frac{1}{2\kappa_0^2} R - \mu_0^2 (H^* H) + a_0 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + b_0 R^2 + c_0 {}^*R^*R + d_0 R;_\mu{}^\mu + e_0 + \frac{1}{4} G_{\mu\nu}^i G^{\mu\nu i} + \frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + |D_\mu H|^2 - \xi_0 R |H|^2 + \lambda_0 (H^* H)^2 \Big] + O\Big(\frac{1}{m_0^2}\Big), \tag{8}$$

where $G_{\mu\nu}^i$, $F_{\mu\nu}^\alpha$, and $B_{\mu\nu}$ are the field strengths of the gauge fields V_μ^i, A_μ^α, and B_μ, respectively. $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and ${}^*R^*R$ is the Euler characteristic, and

$$\mu_0^2 = \frac{4}{3\kappa_0^2}, \quad a_0 = -\frac{9}{8 g_{03}^2}, \quad b_0 = 0, \tag{9}$$

$$c_0 = -\frac{11}{18} a_0, \quad d_0 = -\frac{2}{3} a_0, \quad e_0 = \frac{45}{4\pi^2} f_0 m_0^4,$$

$$\lambda_0 = \frac{4}{3} g_{03}^2 \frac{z^2}{y^4}, \quad \xi_0 = \frac{1}{6}.$$

We have also denoted

$$y^2 = \mathrm{Tr}(|k_0^d|^2 + |k_0^u|^2 + \tfrac{1}{3}|k_0^e|^2),$$

$$z^2 = \mathrm{Tr}(|k_0^d|^4 + |k_0^u|^4 + \tfrac{1}{3}|k_0^e|^4),$$

$$D_\mu H = \partial_\mu H - \tfrac{i}{2} g_{02} A_\mu^\alpha \sigma^\alpha H - \tfrac{i}{2} g_{01} B_\mu H.$$

The Einstein, Yang-Mills, and Higgs terms are normalized by taking

$$\frac{15 m_0^2 f_2}{4\pi^2} = \frac{1}{\kappa_0^2}, \quad \frac{g_{03}^2 f_4}{\pi^2} = 1, \quad g_{03}^2 = g_{02}^2 = \frac{5}{3} g_{01}^2. \tag{10}$$

Relations (10) among the gauge coupling constants coincide with those coming from SU(5) unification.

We shall adopt Wilson's viewpoint of the renormalization group approach to field theory [9] where the spectral action is taken to give the *bare* action with bare quantities $a_0, b_0, c_0, \ldots$ and at a cutoff scale Λ which regularizes the action the theory is assumed to take a geometrical form. We have included the b_0 term because the renormalized theory would require such a term with the $b_0 = 0$ taken as a boundary condition. The perturbative expansion is then reexpressed in terms of *renormalized* physical quantities. The fields also receive wave function renormalization.

The renormalized action receives counterterms of the same form as the bare action but with physical parameters, $a, b, c, \ldots$. The renormalization group equations will yield relations between the bare quantities and the physical quantities with the addition of the cutoff scale Λ. Conditions on the bare quantities would translate into conditions on the physical quantities. The renormalization group equations of this system were studied by Fradkin and Tseytlin [10] and are known to be renormalizable, but nonunitary [11] due to the presence of spin-two ghost (tachyon) pole near the Planck mass. We shall not worry about nonunitarity (see, however, Ref. [12]), because in our view at the Planck energy the manifold structure of space-time will break down and must be replaced with a genuinely noncommutative structure.

Relations between the bare gauge coupling constants as well as Eqs. (9) have to be imposed as boundary conditions on the renormalization group equations [9,13]. The bare mass of the Higgs field is related to the bare value of Newton's constant, and both have quadratic divergences in the limit of infinite cutoff Λ. The relations between m_0^2, e_0, and the physical quantities are

$$m_0^2 = m^2\left[1 + \frac{(\Lambda_2/m_2 - 1)}{32\pi^2}\left(\frac{9}{4}g_2^2 + \frac{3}{4}g_1^2 + 6\lambda - 6k_t^2\right)\right] + O(\ln\frac{\Lambda^2}{m^2}) + \dots,$$

$$e_0 = e + \frac{\Lambda^4}{32\pi^2}(62) + \dots. \tag{11}$$

For $m^2(\Lambda)$ to be small at low energies m_0^2 should be tuned to be proportional to the cutoff scale according to Eq. (11). Similarly the bare cosmological constant is related to the physical one (which must be tuned to zero at low energies). There is also a relation between the bare scale κ_0^{-2} and the physical one κ^{-2} which is similar to Eq. (11) (but with all one-loop contributions coming with the same sign) which shows that $\kappa_0^{-1} \sim m_0$ and Λ are of the same order as the Planck mass. The renormalization group equations for the gauge coupling constants g_1, g_2, and g_3 are the same as those with SU(5) boundary conditions, and can be easily solved using the present experimental values for $\alpha_{\rm em}^{-1}(M_z)$ and $\alpha_3(M_z)$ to give [14]

$$\Lambda \simeq 10^{15}\ (\text{Gev}), \quad \sin^2\theta_w \simeq 0.210. \tag{12}$$

There is one further relation in our theory between the $\lambda_0(H^*H)^2$ coupling and the gauge couplings in Eq. (9) imposed at the scale Λ. This relation could be simplified if we assume that the top quark Yukawa coupling is much larger than all the other Yukawa couplings. In this case the equation simplifies to $\lambda(\Lambda) = (16\pi/3)\alpha_3(\Lambda)$. Therefore the value of λ at the unification scale is $\lambda_0 \simeq 0.402$, showing that one does not go outside the perturbation domain. In reality the renormalization group equations for λ and k_t together with the boundary condition on λ could be used to determine the Higgs mass at the low-energy scale M_z. The renormalization group equations are complicated and must be integrated numerically [15]. One can read the solution for the Higgs mass from the study of the triviality bound, and this gives $m_H = 170-180$ Gev. One expects this prediction to be correct to the same order as that of $\sin^2\theta_w$ which is off from the experimental value by 10%. Therefore the bare action we obtained and associated with the spectrum of the standard model is consistent within 10% provided the cutoff scale is taken to be $\Lambda \sim 10^{15}$ Gev at which the action becomes geometrical.

There is, however, a stronger disagreement where Newton's constant comes out to be too large. This is so because the gravity sector requires the cutoff scale to be of the same order as the Planck scale while the condition on gauge coupling constants gives $\Lambda \sim 10^{15}$ Gev. One easy way to avoid this problem is to assume that the spectrum contains in addition a fermionic particle which only interacts gravitationally (such as a gravitino), but at present we shall not commit ourselves. These results must be taken as an indication that the spectrum of the standard model has to be altered as we climb up in energy. The change may at low energies (just as in supersymmetry which also pushes the cutoff scale to 10^{16} Gev) or at some intermediate scale. Ultimately one would hope that modification of the spectrum will increase the cutoff scale nearer to the Planck mass as dictated by gravity.

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