Large Quantum Gravity Effects: Unforeseen Limitations of the Classical Theory

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(Received 8 October 1996)

Three-dimensional gravity coupled to Maxwell (or Klein-Gordon) fields is exactly soluble under the assumption of axisymmetry. The solution is used to probe several quantum gravity issues. In particular, it is found that if there is an electromagnetic wave of Planckian frequency even with such low amplitude that the curvature of the classical solution is small, the uncertainty in the quantum metric can be very large. More generally, the quantum fluctuations in the geometry are large unless the number and frequency of photons satisfy the inequality $\mathcal{N}(\hbar G\omega)^2 \ll 1$. Results hold also for a sector of the four-dimensional theory (consisting of Einstein-Rosen gravitational waves). [S0031-9007(96)01849-2]

PACS numbers: 04.60.Ds, 04.60.Kz, 04.62.+v

This work has several motivations. The first stems from the fact that recently there has been considerable mathematical progress in various approaches to quantum gravity. Hence, it is now important to devise criteria for the physical viability of potential solutions. One obvious demand is that the theory should admit "a sufficient number of" semiclassical states. However, the precise meaning of this requirement is not obvious. What exactly is the set of classical solutions that should admit quantum analogs? A natural strategy is to try to develop intuition by analyzing exactly soluble models. Such models can also provide insights into a number of other issues. For example, since the modes that are significant at infinity in the Hawking process have trans-Planckian frequencies near the horizon, there has been considerable interest in the role of high frequency fields in quantum gravity. There have been suggestions that an adequate description of such modes may violate local Lorentz invariance [1]. It has also been suggested that the usual counting of states at high frequencies is inadequate [2] and that the correct counting may lead to a "holographic picture" in which physics within a region can be captured by states residing at its boundary [3]. Finally, there exists a mathematically well-defined theory called semiclassical gravity in which the gravitational field is treated classically but the matter fields are quantized [4]. One hopes that this theory would be phenomenologically satisfactory in the low energy regime. Is this indeed the case? Can one derive this theory from full quantum gravity? What does it "miss out?"

The purpose of this Letter is to analyze such issues in the context of an exactly soluble model: three-dimensional (Lorentzian) gravity coupled to Maxwell fields, restricted by the condition that there be a rotational symmetry. As is well known, in three dimensions, a Maxwell field F_{ab} is dual to a scalar field Φ via $F_{ab} = \epsilon_{ab}{}^{c}\nabla_c\Phi$ and Maxwell equations reduce to the wave equation, $g^{ab}\nabla_a\nabla_b\Phi = 0$. Furthermore, the duality map sends the stress-energy tensor $T_{ab}(F)$ to $T_{ab}(\Phi)$. Therefore, mathematically, the Einstein-Maxwell system (without further fields and interactions) is equivalent to the Einstein-Klein-Gordon system. For simplicity, we will use the scalar field formulation. Since this system is exactly soluble under the assumption of axisymmetry [5–7], we can explicitly analyze the status of various quantum issues in the solution. We will find that quantum effects can be large in unforeseen situations so that a very large number of solutions of the classical theory are spurious: they do not appear in the classical limit of the quantum theory. It should not be difficult to extend the discussion to the treatment of Callen-Giddings-Harvey-Strominger (CGHS) dilatonic black holes given in [8].

In order to compare and contrast various effects and better understand their origin, let us proceed in steps, "switching $\hbar$ and G on and off" as needed. (However, throughout we set $c = 1$.) Let us begin with quantum field theory without gravity ($G = 0$). Then we just have an axisymmetric quantum Maxwell field in three-dimensional Minkowski space-time (M, η_{ab}), represented by the operator-valued distribution

$$\hat{\Phi}(R,T) = \int_0^\infty d\omega[f_\omega^+(R,T)A(\omega) + f_\omega^-(R,T)A^\dagger(\omega)]. \tag{1}$$

Here $f_\omega^+(R,T) = J_o(\omega R)\exp -i\omega T$ (with J_o, the Bessel function) are the positive frequency solutions, $A^\dagger$ and A, the creation and annihilation operators. In the rest frame used in the expansion (1), the Hamiltonian is given by

$$\hat{H}_o = \int_0^\infty d\omega\, \omega A^\dagger(\omega)A(\omega). \tag{2}$$

The system is Poincaré invariant and other Poincaré generators can be expressed similarly. The Hilbert space is the Fock space. Given *any* classical solution

$$\tilde{C}(R,T) = \int_0^\infty d\omega[f_\omega^+(R,T)C(\omega) + f_\omega^-(R,T)\bar{C}(\omega)], \tag{3}$$

there is a coherent state, $|\Psi_c\rangle$ which remains peaked around $\tilde{C}$ for all times:

$$|\Psi_c\rangle = K_c \exp\frac{1}{\hbar}\int_0^\infty d\omega\, C(\omega)A^\dagger(\omega)\,|0\rangle, \tag{4}$$

where K_c is the normalization constant. However, quantum fluctuations in physical quantities are often negligible compared to the expectation value of that quantity only when the the expected number of photons,

$$\mathcal{N} = \langle\Psi_c|\hat{N}|\Psi_c\rangle \equiv \frac{1}{\hbar}\int d\omega|C(\omega)|^2, \tag{5}$$

is large. For example, we have $(\triangle\hat{H}_o/\langle\hat{H}_o\rangle)^2 \sim 1/\mathcal{N}$, where, as usual, $(\triangle\hat{H}_o)^2 = \langle\hat{H}_o^2\rangle - \langle\hat{H}_o\rangle^2$ is the uncertainty in the value of the (normal ordered) Hamiltonian. The entire discussion is independent of where $C(\omega)$ may be peaked; since there is no preferred scale, the theory cannot distinguish high frequency photons from the low frequency ones.

Let us now switch off $\hbar$ and switch on G; i.e., consider classical general relativity. (Since in three dimensions G has the physical dimensions of inverse mass, we now have a mass scale which features prominently in the description.) The theory is now governed by a set of *coupled, nonlinear* partial differential equations on $M \equiv R^3$:

$$g^{ab}\nabla_a\nabla_b\Phi = 0 \quad\text{and}\quad R_{ab} = 8\pi G\nabla_a\Phi\nabla_b\Phi\,, \tag{6}$$

where R_{ab} is the Ricci tensor of g_{ab}.

However, because of rotational invariance *and* presence of the axis, two simplifications arise [5,9]. First, M admits a global chart T, R, θ (unique up to $T \rightarrow T + \mathrm{const}$) such that the norm of the rotational Killing field $\partial/\partial\theta$ is R^2. In this chart, the metric has the form

$$g_{ab}dx^a dx^b = e^{G\Gamma(R,T)}(-dT^2 + dR^2) + R^2 d\theta^2; \tag{7}$$

there is only one unknown metric coefficient, $\Gamma(R,T)$. The second simplification is that Φ satisfies the wave equation with respect to g_{ab} if and only if it satisfies the wave equation with respect to the flat metric η_{ab} obtained by setting $\Gamma(R,T) = 0$ in (7). Thus, the two equations in (6) can now be *decoupled:* we can first solve for Φ and then attempt to solve Einstein's equation to determine $\Gamma(R,T)$. Given a solution $\tilde{C}(R,T)$ to the wave equation, the corresponding $\Gamma(R,T)$ turns out to be

$$\Gamma(R,T) = \frac{1}{2}\int_0^R dR\, R[(\partial_T\tilde{C})^2 + (\partial_R\tilde{C})^2], \tag{8}$$

which is precisely the (Minkowskian) energy of the scalar field $\tilde{C}(R,T)$ in a box of radius R at time T. Thus, the problem of solving the coupled Einstein-Maxwell system reduces to that of solving the wave equation on the flat space (M, η_{ab}). As in the previous case (with $\hbar \neq 0, G = 0$), there is no natural frequency scale and the general description in insensitive to the detailed profile of $C(\omega)$. The asymptotic metric, for example, depends only on the total (Minkowskian) energy $H_0(\tilde{C})$ and not to where $C(\omega)$ is peaked.

Suppose the field $\tilde{C}(R,T)$ has initial data of compact support. Then, a neighborhood of infinity is source-free and *in that region* the metric is locally flat, with $\Gamma(R,T) = H_o(\tilde{C})$, a constant (given by the total energy of $\tilde{C}$). Nonetheless, when $GH_o(\tilde{C}) \geq 1$, the asymptotic geometry is *quite different* from that of the globally flat metric η_{ab} (where $\Gamma = 0$). In particular, there is a deficit angle $2\pi[1 - \exp(-GH_o/2)]$ at infinity [10]. As a consequence, *we no longer have Poincaré invariance even at infinity* [10–12]. There is a preferred rest frame given by $\partial/\partial T$ selected by the total energy momentum of the system. Perhaps the most dramatic difference from the Minkowskian situation is that the total Hamiltonian of the system (including gravity) is now bounded above [11,12]: it is given by

$$H = \frac{1}{4G}(1 - e^{-4GH_o})\,. \tag{9}$$

In the weak field limit (i.e., as $GH_o(\tilde{C}) \rightarrow 0$), H does tend to H_o. However, in the strong field regime, the general relativistic corrections dominate and the total Hamiltonian is a bounded, *nonpolynomial* functional of H_o.

Let us now switch on both G and $\hbar$, i.e., consider quantum gravity proper. [Now, we have a natural mass scale (G^{-1}) as well as a natural length scale $(G\hbar)$.] As one might expect, the model can be quantized exactly [6,7,13]. In a Hamiltonian framework, one can first quantize the field Φ exactly as in Minkowski space and then define the metric operator by suitably regularizing the right side of (8). As in any nontrivial model which is solved by mapping it to a trivial model, the nontrivialities lie in the *relation* between the two models. In the present case, the physical model leads us to consider, in particular, operators corresponding to $\exp[G\Gamma(R,T)]$ and study the resulting quantum geometry which exhibits several interesting features [7].

For simplicity, here we will focus our attention on the asymptotic form of the metric [whose only nontrivial component is $g_{RR} = -g_{TT} = \exp(GH_o)$] and on the Hamiltonian H. The corresponding operators will be taken to be

$$\hat{g}_{RR} = e^{G\hat{H}_o} \quad\text{and}\quad \hat{H} = \frac{1}{4G}(1 - e^{-4G\hat{H}_o})\,, \tag{10}$$

which are both positive definite and self-adjoint.

The question now is: Are there semiclassical states which approximate solutions $[\tilde{C}(R,T), g_{ab}]$ of Einstein-Maxwell equations throughout the space-time? Since we wish to approximate, in particular, the solution $\tilde{C}$ to the wave equation, the obvious candidates are, again, the coherent states $|\Psi_c\rangle$. Let us compute the expectation values of operators of interest and see if they give us back the classical fields. Fortunately, these expectation values as well as fluctuations around them can be expressed in closed forms. We have

$$\langle\Psi_c|\hat{\Phi}(R,T)|\Psi_c\rangle = \tilde{C}(R,T)\,;$$

$$\langle\Psi_c|\hat{g}_{RR}|\Psi_c\rangle = e^{\frac{1}{\hbar}\int d\omega|C(\omega)|^2(e^{G\hbar\omega}-1)}; \tag{11}$$

$$\langle\Psi_c|\hat{H}|\Psi_c\rangle = \frac{1}{4G}\left[1 - e^{\frac{1}{\hbar}\int d\omega|C(\omega)|^2(e^{-4G\hbar\omega}-1)}\right].$$

Thus, for the scalar field, of course, we recover the classical field. However, for the metric and the Hamiltonian, the expectation values are not simply related to the corresponding classical expressions:

$$g_{RR} = e^{G \int d\omega\, \omega |C(\omega)|^2}, \quad H = \frac{1}{4G}(1 - e^{-4G \int d\omega\, \omega |C(\omega)|^2}). \tag{12}$$

(In particular, the expectation values have an explicit $\hbar$ dependence.)

However, since we now have a natural length scale, we can analyze limits. In the low frequency limit, $G\hbar\omega \ll 1$, the $\hbar$ dependence disappears in the leading order and we recover the classical values *provided the frequency is so small that* $\mathcal{N}(G\hbar\omega)^2 \ll 1$. In this case, we can also evaluate the quantum corrections systematically to any desired order:

$$\langle\Psi_c|\hat{g}_{RR}|\Psi_c\rangle = g_{RR}\left[1 + \frac{1}{2}\mathcal{N}(G\hbar\omega_o)^2 + \dots\right],$$
$$\langle\Psi_c|\hat{H}|\Psi_c\rangle = H(\tilde{C}) - \frac{4}{G}\mathcal{N}(G\hbar\omega_o)^2 e^{-4GH_o(\tilde{C})} + \dots, \tag{13}$$

where in the last terms, we have evaluated the integrals approximately for the case when $C(\omega)$ is sharply peaked at a frequency ω_o. At high frequencies, $G\hbar\omega \gg 1$, there is always a significant disagreement with the classical expressions:

$$\langle\Psi_c|\hat{g}_{RR}|\Psi_c\rangle = e^{\frac{1}{\hbar}\int d\omega |C(\omega)|^2 e^{G\hbar\omega}} \approx e^{\mathcal{N}(e^{G\hbar\omega_o})},$$
$$\langle\Psi_c|\hat{H}|\Psi_c\rangle = \frac{1}{4G}[1 - e^{-\frac{1}{\hbar}\int d\omega |C(\omega)|^2}],$$
$$\approx \frac{1}{4G}[1 - e^{-\mathcal{N}}]. \tag{14}$$

Note that now $\hbar$ does not disappear even in the leading order terms and the departures from the classical values occur even in the *asymptotic* metric.

Let us analyze the quantum uncertainties. For brevity, we will now focus just on the metric operator. The exact expression turns out to be:

$$\left(\frac{\Delta\hat{g}_{RR}}{\langle\hat{g}_{RR}\rangle}\right)^2 = [e^{\frac{1}{\hbar}\int d\omega |C(\omega)|^2 (1 - e^{G\hbar\omega})^2} - 1]. \tag{15}$$

In the low frequency regime, $G\hbar\omega \ll 1$, we therefore have

$$\left(\frac{\Delta\hat{g}_{RR}}{\langle\hat{g}_{RR}\rangle}\right)^2 \approx e^{\mathcal{N}(G\hbar\omega_o)^2} - 1, \tag{16}$$

and, in the high frequency regime, $G\hbar\omega \gg 1$, we have

$$\left(\frac{\Delta\hat{g}_{RR}}{\langle\hat{g}_{RR}\rangle}\right)^2 \approx e^{\mathcal{N}(e^{2G\hbar\omega_o})}. \tag{17}$$

Thus, in the high frequency regime, the quantum fluctuations in the metric are huge even when the state is sharply peaked at the given classical scalar field $\tilde{C}$ and they become larger as the number $\mathcal{N}$ of expected photons increases. (Recall that the opposite holds for the $\hat{\Phi}$ field.) Suppose the amplitude $C(\omega)$ is zero everywhere except for a "blip" near Planck frequency with $\int d\omega |C|^2 \sim \hbar$. Then $\mathcal{N} \sim 1$ and the space-time curvature in the classical solution is small. Nonetheless, Eq. (15) implies that $(\triangle\hat{g}/\langle\hat{g}\rangle)^2 > 10$. (Quantum uncertainties can also be large if $C(\omega)$ has a large peak in the low frequency regime and a blip at a (trans-)Planckian frequency.) Next, consider the low frequency regime. Now, for the quantum uncertainties in the Maxwell field to be negligible, we need $\mathcal{N} \gg 1$ *and* for the uncertainties in the metric to be negligible, we need the frequency to be so low that $\mathcal{N}(G\hbar\omega)^2 \ll 1$. Only in this regime does the classical solution $(\tilde{C}, g_{ab})$ approximate the predictions of the quantum theory.

To summarize, the quantum theory does have infinitely many states with semiclassical behavior. This by itself is a nontrivial property, given the nonlinearities of the Einstein equations. However, all these states are peaked at a very restricted class of classical solutions $(\tilde{C}, g_{ab})$: the Maxwell field profiles $C(\omega)$ have to satisfy the two inequalities: $\mathcal{N} \gg 1$ and $\mathcal{N}(G\hbar\omega)^2 \ll 1$. With hindsight, the second condition can be understood intuitively as follows [14]. From Eq. (10) we can conclude that the fluctuations in $\hat{g}_{RR}$ are large whenever $G(\Delta\hat{H}_o) \geq 1$. But we know that $\Delta\hat{H}_o \sim \hbar\omega\sqrt{N}$ [see the remark below Eq. (5)]. Hence, the fluctuations in $\hat{g}_{RR}$ are large unless $\mathcal{N}(G\hbar\omega)^2 \ll 1$. As a consequence, many apparently "tame" classical solutions fail to arise in the classical limit of the quantum theory; they fail to serve as leading order approximations even in regions where space-time curvature is small. This is a subtle failure of the classical theory which has been overlooked so far. Nonlinearities of Einstein's equations magnify the small quantum uncertainties in the Maxwell field to huge fluctuations in the metric.

We will conclude by summarizing some ramifications of these results.

(1) *Semiclassical gravity.*—Recall [4] that a solution to semiclassical gravity is a quadruplet $(M, g_{ab}, \hat{\Phi}, |\Psi\rangle)$ consisting of a metric g_{ab} (of hyperbolic signature) on a manifold M, a quantum matter field $\hat{\Phi}$ satisfying a field equation on (M, g_{ab}), and a state $|\Psi\rangle$ of the field such that the semiclassical Einstein equation $G_{ab} = 8\pi G\langle\Psi|\hat{T}_{ab}|\Psi\rangle$ is satisfied. In four dimensions, it is difficult to find exact solutions to this theory and several important issues about the nature of the approximation involved remain open. What is the situation in the present model? Now, it is straightforward to show that the theory admits an infinite number of solutions. Indeed, given *any* (axisymmetric) classical solution $(\tilde{C}, g_{ab})$ to Einstein-Maxwell equations on $M = R^3$, the quadruplet $(M, g_{ab}, \hat{\Phi}, |\Psi_c\rangle)$ is an exact solution to semiclassical gravity. While this abundance serves to show that the theory is mathematically rich, it also brings out its physical limitations. For, unless the profiles $C(\omega)$ of the Maxwell field satisfy $\mathcal{N}(G\hbar\omega)^2 \ll 1$, the solution to

semiclassical gravity is spurious; it does not approximate the situation in the full theory. Very roughly, it knows about both G and $\hbar$ separately, but does not fully exploit the Planck length which requires *both* at once.

(2) *High frequency behavior*.—We saw that something peculiar does happen at (trans-)Planckian frequencies. However, *local* Lorentz invariance is not broken; the field equation governing $\tilde{C}$ does not, for example, involve an additional vector field. The frequency refers to the asymptotic rest frame $(\partial/\partial T)$. The relativistic dispersion relation is not modified, nor is there a high frequency cutoff. Similarly, within this model, there is no hint of the behavior suggested by the holographic hypothesis. It is sometimes conjectured that in a full quantum theory the trans-Planckian strangeness will trickle down and even affect the vacuum since the vacuum fluctuations involve all modes. There is no evidence of such a behavior either. The vacuum $|0\rangle$ is stable although the vacuum fluctuations involve modes of arbitrarily high frequencies. Furthermore, the vacuum is an eigenstate both of $\hat{H}$ (with eigenvalue zero) and of $\hat{g}_{RR} = -\hat{g}_{TT}$ (with eigenvalue one). The peculiarity that does arise is of a rather different nature: *classical geometry simply fails to be a good approximation* when $C(\omega)$ has a significantly high frequency component and one has to take the quantum nature of geometry—with all its fluctuations—seriously.

(3) *Three versus four dimensions*.—Since Newton's constant in three dimensions has physical dimensions of inverse mass (rather than length times inverse mass) we cannot associate a Schwarzschild radius to a given mass (without introducing a cosmological constant). This explains intuitively why we did not encounter gravitational collapse. If we consider a spherical scalar field in four dimensions, on the other hand, the theory has two sectors: one leading to black holes (strong data) and the other leading just to scattering (weak data). The present model can only give us insights into the second sector. Hence, it would be imprudent to draw from it definitive conclusions about the Bekenstein bound [2] and related conjectures [3]. A second difference is that (in the asymptotically flat context) while in three dimensions the metric outside sources is flat, in four-dimensional general relativity it falls off as $1/r$. Therefore, in four-dimensional asymptotically flat situations, the effect of trans-Planckian frequencies will decay and the fluctuations will be significant only near (but possibly in a macroscopic region around) the sources. The model does, nonetheless, provide concrete evidence for large quantum gravity effects in low-curvature regimes that have been generally ignored. Furthermore, they are not artifacts of an unusual quantization procedure; we used only traditional Fock-space methods. Therefore, with appropriate modifications, features encountered here should persist in more sophisticated models such as a spherically symmetric scalar field coupled to gravity in $3 + 1$ dimensions. Indeed, our model is completely equivalent [5] to a "midisuperspace" consisting of Einstein-Rosen waves in *four-dimensional* source-free general relativity. Hence, our results apply directly to this sector of four-dimensional quantum gravity [13]. Similar reductions exist also in string theory [15].

Details as well as several other issues will be discussed in [16].

I thank Chris Beetle, Eanna Flanagan, Donald Marolf, Monica Pierri, and Thomas Thiemann for discussions. This work was supported in part by NSF Grant No. 95-14240, by the Eberly research funds of Penn State University, and by the Erwin Schrödinger International Institute for Mathematical Sciences, Vienna.

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