Geometrical Resonance as a Chaos Eliminating Mechanism

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The notion of geometrical resonance is introduced as a natural, fully nonlinear, generalization of the usual or frequency resonance. The geometrical resonance is shown to provide the mechanism underlying the so-called nonfeedback control of chaos by means of an almost adiabatic invariant associated with each geometrical resonance solution. [S0031-9007(96)00576-5]

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The notion of resonance (nonlinear resonance) has always been identified with how well the driving period T_d fits (a rational fraction of) a natural period T_0 of the underlying conservative system. In this Letter, the aim is to extend this linear-system-based notion of resonance to a fully nonlinear formulation based on a local energy conservation requirement, considering the one-dimensional, damped, and nonautonomous, nonlinear oscillator

$$\ddot{x} - g(x) = -d(x,\dot{x}) + p(x,\dot{x})F(t), \tag{1}$$

where $g(x) \equiv -\partial V/\partial x$ [$V(x)$ being an arbitrary time-independent potential], $-d(x,\dot{x})$ is the damping force, and $p(x,\dot{x})F(t)$ is a general temporal modulation. The basic idea is that the amplitude, period, and shape of $F(t)$ must be such as to preserve a previously chosen natural response from the underlying conservative system; this will be called *geometrical resonance* (GR), because the shape driving is just as meaningful as the period for the completely nonlinear problem [1]. Notice that to take the shape into account is equivalent to considering at once *all* the nonlinear resonances (periods) suitably weighted. In general, if $x_{\mathrm{GR}}(t)$ is a GR solution of Eq. (1), it must satisfy

$$-d(x_{\mathrm{GR}},\dot{x}_{\mathrm{GR}}) + p(x_{\mathrm{GR}},\dot{x}_{\mathrm{GR}})F_{\mathrm{GR}}(t) = 0. \tag{2}$$

This is equivalent to the (local) energy conservation requirement $(1/2)\dot{x}_{\mathrm{GR}}^2(t) + V[x_{\mathrm{GR}}(t)] = \text{const}$. The period, shape, and amplitude of $F_{\mathrm{GR}}(t)$ will be determined by those of $x_{\mathrm{GR}}(t)$. Obviously, in the fully linear limit we recover the usual resonance requirement [$F_{\mathrm{GR}}(t) = -\text{const} \times \dot{x}_{\mathrm{GR}}(t)$, i.e., $F_{\mathrm{GR}}(t)$ harmonic and $T_d = T_0$]. The GR for nonautonomous Hamiltonian systems will be considered elsewhere.

The GR provides the mechanism underlying the so-called nonfeedback control technique [2–4]. This method of suppressing chaos works by adding a weak external periodic forcing or perturbing a system parameter by small harmonic perturbations to the initially chaotic system. There have been several theoretical [2], numerical [3], and experimental investigations [4] of nonfeedback control. As an illustration of the use of this concept, consider the general [and widely used (cf. Refs. [3,4])] system

$$\ddot{x} - g(x) = -\gamma\dot{x} + F_c \mathrm{har}\left(\frac{2\pi t}{T}\right) + F_{nc}\mathrm{har}\left[\frac{2\pi q t}{Tp} + \phi\right], \tag{3}$$

where the notation $\mathrm{har}(x)$ means indistinctly $\sin(x)$ or $\cos(x)$, and p, q are relatively prime integers. When the suppressory driving term is absent ($F_{nc} \equiv 0$), we assume that the system is in a chaotic state for a certain damping γ and forcing F_c, and for a *given* initial condition. Now, the necessary and sufficient condition to be verified by the total driving force in order for the system (3) to be found in a GR may be written [cf. Eq. (2)]

$$\dot{x}_{\mathrm{GR}} = \left(\frac{1}{\gamma}\right)\left\{F_c\,\mathrm{har}\left(\frac{2\pi t}{T}\right) + F_{nc}\,\mathrm{har}\left[\frac{2\pi q t}{Tp} + \phi\right]\right\}, \tag{4}$$

where $x_{\mathrm{GR}}(t)$ is a (T'-periodic) response—based on the same aforementioned initial condition—of the underlying conservative system. Generally, $x_{\mathrm{GR}}(t)$ will be a nonlinear periodic response [5], and so we can write

$$\dot{x}_{\mathrm{GR}}(t) = \sum_{n=1}^{\infty} a_n\,\mathrm{har}\left(\frac{2\pi n t}{T'} + \varphi_n'\right). \tag{5}$$

Clearly there cannot exist an added harmonic suppressory driving force exactly satisfying the GR condition [Eq. (4)]. However, we can find the optimal values of F_{nc}, ϕ, and (p/q) which *most closely* preserve the energy in the following sense. Let us assume that for the optimal choice (and the same initial condition) the corresponding actual solution $x(t)$ remains—after the transient—close to the GR solution: $x(t) = x_{\mathrm{GR}}(t) + \delta x(t)$, where $\delta x(t)$ is a small deviation with $d(\delta x)/dt \ll \delta x/T'$. Then, one conjectures that the ratio of the energy to the frequency $1/T'$ is a *local almost adiabatic invariant* [6] to lowest order in δx, i.e.,

$$\left\langle \frac{d}{dt}\left(\frac{E}{1/T'}\right)\right\rangle_{T'} \equiv \int_0^{T'}\left(\frac{dE}{dt}\right)dt \simeq 0. \tag{6}$$

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A particular case is studied *numerically* by Qu *et al.* [3]:

$$\ddot{x} + x^3 = -\gamma\dot{x} + F_c \cos\left(\frac{2\pi t}{T}\right) + \alpha F_c \cos\left[\frac{2\pi q t}{Tp} + \phi\right], \qquad (7)$$

where $\alpha \equiv F_{nc}/F_c$. *Analytically*, the T'-periodic solutions of the conservative system are

$$x(t) = A\,\mathrm{cn}[4K(m=1/2)t/T' + \phi'; m=1/2],$$
$$\dot{x}(t) = -A^2\,\mathrm{sn}[4K(m=1/2)t/T' + \phi'; m=1/2] \times \mathrm{dn}[4K(m=1/2)t/T' + \phi'; m=1/2], \qquad (8)$$

where $\mathrm{sn}(u;m)$, $\mathrm{cn}(u;m)$, and $\mathrm{dn}(u;m)$ are Jacobian elliptic functions of parameter m, $K(m)$ is the complete elliptic integral of the first kind [7], and $A = 4K(m = 1/2)/T'$. From the assumption $x(t) = x_{\mathrm{GR}}(t) + \delta x(t)$, the energy is given (to first order in δx) by $E = E_{\mathrm{GR}} + [\dot{x}_{\mathrm{GR}} + x_{\mathrm{GR}}^3]\delta x$, and thus

$$\left\langle \frac{dE}{dt} \right\rangle_{T'} = -\gamma\langle \dot{x}_{\mathrm{GR}}^2\rangle_{T'} + F_c\left\langle \dot{x}_{\mathrm{GR}} \cos\left(\frac{2\pi t}{T}\right)\right\rangle_{T'} + \alpha F_c\left\langle \dot{x}_{\mathrm{GR}} \cos\left[\frac{2\pi q t}{Tp} + \phi\right]\right\rangle_{T'} + O(\delta x). \qquad (9)$$

The integrals can be evaluated from standard integral tables [8]. Finally, from the almost adiabatic invariant condition [Eq. (6)], we obtain

$$\alpha \approx \left[\frac{1}{\sin(\varphi_{n^*} - \phi)}\right]\left\{\frac{16K^4(m=1/2)}{3\pi^2}\left(\frac{\gamma}{F_c}\right)\left(\frac{q}{p}\right)^2 \times [(2n^*+1)^2 T^2 a_{n^*}(m=1/2)]^{-1} - \sin\varphi_{n^*} - 2\left[\frac{\sin(\pi T'/T)}{\pi T'}\right] S(p,q,n^*)\right\}, \qquad (10)$$

with

$$S(p,q,n^*) \equiv \sum_{\substack{n=0 \\ n\neq n^*}}^{\infty} \frac{a_n(m=1/2)}{a_{n^*}(m=1/2)} \times \left[\frac{2n+1}{(2n+1)^2 - p^2(2n^*+1)^2/q^2}\right] \times \sin\left[\frac{(2n^*+1)p\pi}{q} + \varphi_n\right], \qquad (11)$$

$$\varphi_n \equiv \frac{(2n+1)\pi\phi'}{2K(m=1/2)}, \qquad (12)$$

$$T' \equiv \frac{(2n^*+1)pT}{q}, \qquad (13)$$

$$a_n(m=1/2) \equiv \frac{(n+1)q^{(n+1/2)}(m=1/2)}{1+q^{n+1}(m=1/2)}, \qquad (14)$$

where $q(m)$ is the nome [7] of parameter m. To control chaos, one desires the control driving term to have an amplitude which is very *small* in comparison with that of the induced-chaos driving, so the optimal values of ϕ must verify $\sin(\varphi_{n^*} - \phi) = \pm 1$, i.e. [cf. Eq. (12)],

$$\phi = \left[\frac{(2n^*+1)\pi\phi'}{2K(m=1/2)} \pm \frac{\pi}{2}\right](\mathrm{mod}\,2\pi). \qquad (15)$$

It is then obvious that the phase difference ϕ between the two forces plays a fundamental role in nonfeedback control, as was found numerically by Qu *et al.* [3] for the system (7), and as was first analytically demonstrated through Melnikov analysis in the two papers of Ref. [2]. From Eq. (15), we see that ϕ has *sensitive dependence on the initial conditions* (through ϕ'), as is observed in numerical experiments, cf. Refs. [2,3]. For a given T, Eq. (13) gives the period T', depending on the approximation (n^*, p, q) we are using, and hence the energy (i.e., A) of the underlying GR response. We can test the predicted α value [Eq. (10)] theoretically by considering the limiting case $\gamma = 0$ (no damping) together with the main resonance ($q = p, n^* = 0$). From Eq. (10), one has that $\alpha \approx -\cos\phi$ and then, for $\phi = 0$, one recovers the corresponding expected [cf. Eq. (7)] periodic solution of $\ddot{x} + x^3 = 0$.

Figure 1 shows the characteristic structure of *tongues* in the $|\alpha|$-$2\phi/\pi$ plane [cf. Eqs. (10), (12), (15)], for several resonances p/q (and so different GRs) and $n^* = 0$, $\gamma = 0.3$, $F_c = 8.85$, $T = 2\pi$. The widths in $2\phi/\pi$ of the various tongues where p/q is fixed increase with $|\alpha|$.

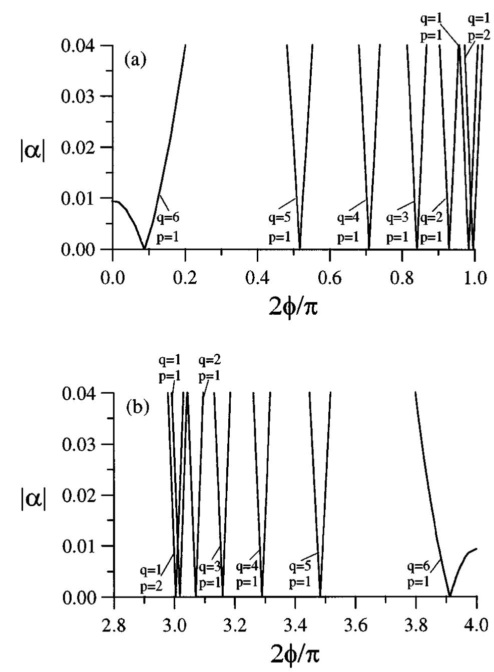

FIG. 1. $|\alpha|$ vs $2\phi/\pi$ [Eqs. (10), (12), (15)] for $n^* = 0$, $\gamma = 0.3$, $F_c = 8.85$, $T = 2\pi$, and different resonances p/q. (a) Mode-locked regions (tongues) for $\phi \in [0, \pi/2]$. (b) Corresponding tongues for the same resonances as in (a) and $\phi \in [3\pi/2, 2\pi]$. Note that there cannot exist tongues over the range $\phi \in]\pi/2, 3\pi/2[$.

The resemblance to the Arnold tongues [9] is meaningful: it is straightforward to show that, given any resonance $p/q > \text{const} > 0$, there exists a tongue given by $\phi = \phi(p/q)$ [cf. Eq. (10)], and also that for fixed $|\alpha|$ the width of a tongue increases if the denominator q in the corresponding resonance p/q increases. Thus, the motion should be phase locked [cf. Eq. (13)] inside the tongues, and chaotic outside them, as is, in fact, observed in numerical experiments. Figure 2 provides an illustrative example for the resonance $p/q = 1/3$ with the other parameters as in Fig. 1. Notice that the boundaries of the tongues obtained numerically [Fig. 2(a)] and theoretically [Fig. 2(b)] correspond very closely in the positions of their minima. The numerical tongues are wider than those from the almost adiabatic invariance, partly because the dots in Fig. 2(a) represent regular motions with periods ≤ 12, and partly because of the perturbative nature of the theoretical approach. Similar agreement between the numerical results and theoretical predictions is found for other resonances (p/q). It is worth mentioning that Azevedo and Rezende [4] experimentally found a similar structure of tongues in a microwave-pumped spin-wave-instability experiment.

Qu *et al.* [3] provided *numerical* evidence of a new type of intermittency, characterized by periodic appearance of regular and chaotic motions, which they call "breathing," when a time-dependent phase difference $\phi(t) = \phi_0 + \Delta\Omega t$ $(\Omega \equiv 2\pi q/Tp,\ \Omega \gg \Delta\Omega)$ is introduced. They indicated phenomenologically that "the dynamics of the new type of intermittency is due to the quasistatic drift in the phase $\phi(t)$; this drift comes from the small detuning." In light of the present theory, the aforementioned breathing effect [3] can be explained as follows: With fixed α, p, and q, let us suppose that, for $\phi(t = 0) = \phi_0$, the initial motion is chaotic, i.e., we are at a point (α, ϕ_0) between two tongues (see Fig. 1), in the α-$2\phi/\pi$ plane. As t increases, so does ϕ—but very slowly—and thus, over the first time interval, the motion evolves chaotically up to an instant t_1^* for which $[\alpha, \phi(t_1^*)]$ is a point close to the boundary of the corresponding (p/q) tongue. For $t > t_1^*$, the motion then evolves under the corresponding local almost adiabatic invariant, and so it cannot be chaotic if the tongue is *wide enough* (depending on α and $\Delta\Omega$). Of course, the motion cannot be periodic (phase locked) over that second time interval, since ϕ does not remain constant. But, as ϕ changes very slowly, the motion will be quasiperiodic up to a second instant t_2^* such that $[\alpha, \phi(t_2^*)]$ is a point close to the opposite boundary of the tongue. For $t > t_2^*$, the motion is again chaotic, and after a time $2\pi/\Delta\Omega$ the motions are qualitatively repeated [the point (α, ϕ) crosses the same tongue in the same direction, during the same time interval $\Delta t \approx t_2^* - t_1^*$]. In a forthcoming paper, I will present a more detailed study, including the application to suppressory parametric perturbations.

In summary, I have introduced the concept of GR (period *plus* shape) as the natural, completely nonlinear, generalization of the common resonance (period). It was shown to provide the dynamics underlying the nonfeedback control of chaos through the conservation of a local almost adiabatic invariant associated to each GR solution. The same ideas were seen to explain a previous, numerically observed, new type of intermittency.

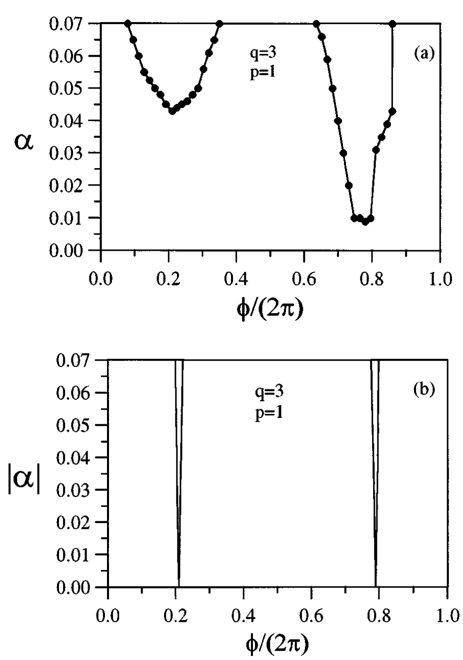

FIG. 2. (a) Boundaries between chaotic and regular motions in the α-$2\phi/\pi$ plane for $\gamma = 0.3$, $F_c = 8.85$, $T = 2\pi$, $q = 3$, $p = 1$, and the initial condition for which the motion is chaotic at $\alpha = 0$. The points represent actual numerical results and are connected by lines to guide the eye. (b) Corresponding boundaries from Eqs. (10), (12), (15) for the same parameters as in (a) and $n^* = 0$.

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[5] In the most unfavorable situation, one can obtain valuable information about the GR solutions from experimental power spectra.
[6] *Local* means that each almost adiabatic invariant is only postulated for (trajectories based on) initial conditions on

a given T'-periodic solution from the associated conservative system. To my knowledge, most of the previous investigations on adiabatic invariants have dealt only with Hamiltonian systems with slowly varying parameters; see, e.g., V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1988), and references therein.

[7] See, e.g., L. M. Milne-Thomson, in *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[8] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1994).
[9] See, e.g., H. G. Schuster, *Deterministic chaos* (VCH, Weinheim, 1989).