

Exact Results for Quantum Phase Transitions in Random XY Spin Chains

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The effect of disorder on the quantum phase transitions induced by a transverse field, anisotropy, and dimerization in XY spin chains is investigated. The low-energy behavior near the critical point is described by a Dirac-type equation for which an exact analytic treatment is possible. Results obtained for the dynamical critical exponent, the specific heat, and the transverse susceptibility agree with results recently obtained using a real space renormalization group decimation technique. A nonzero transverse field changes the universality class of the anisotropy transition. [S0031-9007(96)01787-5]

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Theoretical studies of quantum phase transitions in the presence of quenched (i.e., time-independent) disorder have been stimulated by recent experiments on spin glasses, ^4He absorbed in porous media, and superconductor-insulator transitions in dirty thin films. Compared to thermal phase transitions in disorder-free systems these transitions are poorly understood because many of the theoretical methods (e.g., exact solutions, renormalization group, ϵ expansions) that have been so useful for pure systems are difficult to implement for disordered systems. Yet these phase transitions are associated with particularly rich physics such as large differences between average and typical behavior, new universality classes, logarithmic scaling, Griffiths phases [1] (in which susceptibilities diverge although there are only short-range correlations), and the breakdown of folklore such as “the correlation length is inversely proportional to the energy gap” [2]. This Letter considers a simple exactly soluble model which has many of these interesting properties.

Fisher recently performed an exhaustive study of the effect of randomness on the simplest spin model to undergo a quantum phase transition: the transverse field Ising spin chain [3]. He used a real space renormalization group decimation technique (RSRGDT) [4] which he claims is exact near the critical point. Fisher found the phase diagram (which included Griffiths phases near the critical point), all the critical exponents, and scaling forms for the magnetization and correlation functions in an external field. The latter have never been derived for

the disorder-free case but can be derived in the presence of disorder because distributions become extremely broad near the critical point. The same model was recently studied numerically by Young and Rieger [5], who found results consistent with Fisher. The RSRGDT has also been used to study the effect of disorder on dimerized [6] and anisotropic [7] spin chains, chains with random spin sizes [8], and quantum Potts and clock chains [9]. Possible experimental realizations of random spin chains are quinolonium(TCNQ) $_2$ [10] and $\text{SR}_3\text{CuPt}_{1-x}\text{Ir}_x\text{O}_6$ [11].

This Letter considers quantum phase transitions in XY spin chains in the presence of disorder. The spin chains are mapped onto a fermion model, the continuum limit of which is a Dirac-type equation with random mass, for which *exact* analytic results can be derived. The results agree with those of the RSRGDT [3,6,7], supporting Fisher’s claim that it is exact near the critical point [12]. New results are obtained for the anisotropy transition in a nonzero average transverse field. It is in a different universality class to the Ising transition.

The Hamiltonian to be considered is

$$H = - \sum_{i=1}^L [J_i^x \sigma_i^x \sigma_{i+1}^x + J_i^y \sigma_i^y \sigma_{i+1}^y + h_i \sigma_i^z]. \quad (1)$$

The $\{\sigma_i^\alpha\}$ are Pauli spin matrices, and the interactions J_i^x , J_i^y , and transverse fields h_i may be independent random variables (see Table I) with a Gaussian distribution. The average values will be denoted $\langle J_i^x \rangle \equiv J^x$, $\langle J_i^y \rangle \equiv J^y$, and $\langle h_i \rangle \equiv h$, and will all be assumed to be positive.

TABLE I. Different parameters for the continuum limit of three different transitions. Near the critical point all are described by the low-energy effective Hamiltonian (3). The lattice constant is set to unity. The Fermi wave vector k_F determines whether the problem is commensurate or not. v_F is the Fermi velocity. Δ is a measure of the deviation from criticality. The variance of the random variable equals γ , which enters Eq. (4).

Transition	$\cos k_F$	v_F	Δ	$\Psi(x = n)^\dagger$	Random variable
Ising	1	$ J^x - J^y $	$h - J^x - J^y$	(c_n^\dagger, c_n)	h_i
Anisotropy	$\frac{h}{J^x + J^y}$	$(J^x + J^y) \sin k_F$	$(J^x - J^y) \sin k_F$	$(c_n^\dagger e^{-ink_F}, c_n e^{-ink_F})$	$J_i^x - J_i^y$
Dimerization	0	$2J$	Δ	$(c_n^\dagger i^{-n}, c_n^\dagger i^n)$	$J_i^x = J_i^y$

At zero temperature and in the absence of disorder the model undergoes two distinct quantum phase transitions [13]. Both transitions are second order. The transition at $J^x + J^y = h$ from a paramagnetic to a ferromagnetic phase will be referred to as the *Ising transition* [14]. The transition at $J^x = J^y$ for $h < (J^x + J^y)$ from an Ising ferromagnet with magnetization in the x direction to one with magnetization in the y direction will be referred to as the *anisotropy transition* [7,15]. The *dimerization transition* involves the Hamiltonian (1) with $\langle J_i^x \rangle = \langle J_i^y \rangle = J + (-1)^i \Delta$ and $\langle h_i \rangle = 0$. At $\Delta = 0$ there is a transition from a spin liquid to a gapped phase with long-range topological order [6].

Under a Jordan-Wigner transformation which maps spins onto spinless fermions, (1) becomes [15]

$$H = - \sum_{i=1}^L [(J_i^x + J_i^y)(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + (J_i^x - J_i^y)(c_i^\dagger c_{i+1}^\dagger - c_i c_{i+1}) + h_i(2c_i^\dagger c_i - 1)], \quad (2)$$

where boundary terms have been neglected and c_i^\dagger and c_i denote creation and annihilation operators for a fermion on site i . In the disorder-free case (2) can be diagonalized by a Bogoliubov transformation [13,15].

The continuum limit of (2) must be taken relative to the Fermi wave vector k_F , the wave vector at which the energy gap in the fermionic spectrum occurs for the disorder-free case [13]. This wave vector depends on which transition is being considered (see Table I). The low-energy properties of all the transitions in Table I are described by [16]

$$H = \int dx \Psi^\dagger \left[-i v_F \sigma^3 \frac{\partial}{\partial x} + V(x) \sigma_+ + V(x)^* \sigma_- \right] \Psi, \quad (3)$$

where σ^a ($a = 1, 2, 3$) and $\sigma_\pm \equiv \frac{1}{2}(\sigma^1 \pm \sigma^2)$ are Pauli matrices. $\Psi(x)$ is a spinor, with (modulo a transformation in spinor space) the components given in Table I. $V(x)$ is Gaussian white-noise potential with

$$\langle V(x) \rangle = \Delta, \quad \langle V(x) V(x')^* \rangle = \gamma \delta(x - x'). \quad (4)$$

Δ measures the deviation from the critical point, and $|\Delta|$ is the energy gap in the absence of disorder. Table I lists values of Δ , the Fermi velocity v_F , and the random variables whose variance equals γ , for the Ising, anisotropy, and dimerization transitions. k_F is *commensurate* with the lattice and $V(x)$ is real for the Ising transition ($k_F = 0$) and the anisotropy and dimerization transitions in zero transverse field ($k_F = \pi/2$). It will be seen that the effect of disorder is significantly different for commensurate and incommensurate cases. The continuum limit describes the low-energy properties ($E \ll v_F$) when $v_F \gg \Delta$, $\sqrt{\gamma}$, i.e., arbitrarily close to the critical point and for weak disorder.

It is useful to define an energy D and a dimensionless parameter δ which are measures of the disorder strength

and the deviation from criticality, respectively,

$$D \equiv \frac{\gamma}{v_F}, \quad \delta \equiv \frac{|\Delta|}{D}. \quad (5)$$

Note that for the Ising transition with $J^y = 0$, to leading order in Δ/J^x , the parameter δ defined by Fisher [3] and Young and Rieger [5] is Δ/D .

The advantage of casting the problem in the form of the Hamiltonian (3) is that the latter has been studied extensively previously, and *exact* analytic expressions given for the energy dependence of the disorder-averaged density of states $\langle \rho(E) \rangle$ and the localization length $\lambda(E)$ [17–23]. (Because of the one-dimensionality all the states are localized by the disorder.) The exact results have been found by Fokker-Planck equations [17], supersymmetry [19,20], the replica trick [23], S -matrix summation [18], and the Dyson-Schmidt method [21]. The localization length can be found because in one dimension it is related to the real part of the one-fermion Green's function [19]. The density of states and the localization length can be written in terms of $f'_\delta(u)$, the derivative of a dimensionless function $f_\delta(u)$ given in Table II,

$$\frac{d}{dE} \frac{1}{\lambda(E)} + i\pi\rho(E) = \pi\rho_0 f'_\delta(E/D), \quad (6)$$

where $\rho_0 \equiv 1/\pi v_F$ is the value of the density of states at high energies ($|E| \gg \Delta, D$).

The low-energy ($|E| \ll D$) behavior of the density of states is given in Table II. For the commensurate case the density of states diverges at $E = 0$ for $\delta < 1/2$ and is zero at $E = 0$ for $\delta > 1/2$. These two cases lead to qualitatively very different behavior. In the former case some susceptibilities will diverge as the temperature approaches zero. This corresponds to a Griffiths or weakly ordered phase [1]. Hence for the Ising transition there will be four phases: ferromagnet, weakly ordered ferromagnet, weakly ordered paramagnet, and paramagnet [3]. The full energy dependence of the density of states for various values of δ is shown in Fig. 1.

Distinctly different behavior occurs for the incommensurate case. The density of states is always finite at zero energy. There is a smooth crossover from gapless behavior ($\langle \rho(0) \rangle \sim \rho_0$ for $\delta < 1$) to effectively gapped behavior ($\langle \rho(0) \rangle \ll \rho_0$ for $\delta \gg 1$). Hence the Griffiths phases still exist but no longer have clearly defined boundaries. Similarly, for the commensurate case more than one type of disorder (e.g., the anisotropy transition with both a random transverse field and random anisotropy) removes the singularity in the density of states [17].

The specific heat.—Because the eigenstates of the Hamiltonian (3) are noninteracting fermions the low-temperature behavior of the specific heat and the transverse susceptibility (for the dimerization and anisotropy transitions) follows from the energy dependence of the disorder-averaged density of states [10,24] and are given

TABLE II. Summary of the low-energy behavior of different physical quantities. The commensurate case corresponds to the Ising transition and to the anisotropy and dimerization transitions in zero transverse field. The incommensurate case corresponds to the anisotropy transition in a nonzero transverse field. For the commensurate case qualitatively different behavior occurs at the critical point ($\delta = 0$) and away from it ($\delta \neq 0$). The function $f_\delta(u)$ determines the full energy dependence of the density of states and the localization length [compare Eq. (6)]. $I_0(x)$ and $I_1(x)$ are the zeroth and first order modified Bessel functions, respectively. $\Gamma(\delta)$ is the gamma function. $H_\delta^{(2)}(u)$ is a Hankel function of index δ . $I_{iy}(x)$ is a modified Bessel function with imaginary index. For small x , $\Gamma(x) \sim x^{-1}$, $I_0(x) \sim 1 + x^2/4$, and $I_1(x) \sim x/2$.

Quantity	Symbol	Commensurate		Incommensurate
		$\delta \neq 0$	$\delta = 0$	
Density of states	$\langle \rho(E) \rangle / \rho_0$	$\frac{\delta}{\Gamma(\delta)^2} \left \frac{E}{D} \right ^{2\delta-1}$	$\frac{D}{ E [\ln \frac{E}{D}]^3}$	$\frac{1}{I_0(2\delta)^2}$
Specific heat	$\langle C(T) \rangle$	$\frac{\delta}{\Gamma(\delta)^2} \left(\frac{T}{D} \right)^{2\delta}$	$\frac{1}{[\ln(\frac{T}{D})]^3}$	$\frac{T}{I_0(2\delta)^2}$
Dynamical critical exponent	z	$\frac{1}{2\delta}$	∞	1
Localization length	$\lambda(E)$	$\frac{v_F}{\Delta}$	$\frac{v_F}{D} \ln \left \frac{D}{E} \right $	$\frac{4v_F}{D} \left[1 + \frac{8\delta I_1(2\delta)}{I_0(2\delta)} \right]^{-1}$
Transverse susceptibility	$\langle \chi_{zz}(T) \rangle$	$T^{2\delta-1}$	$\frac{1}{T [\ln(\frac{T}{D})]^2}$	$\frac{1}{I_0(2\delta)^2}$
	$f_\delta(u)$		$-u \frac{\partial}{\partial u} \ln[H_\delta^{(2)}(u)]$	$\delta \frac{\partial}{\partial \delta} \ln[I_{-2iu}(2\delta)]$

in Table II. The transverse susceptibility clearly shows a Griffiths singularity. The results for the commensurate case agree with the RSRGDT [3,6,7].

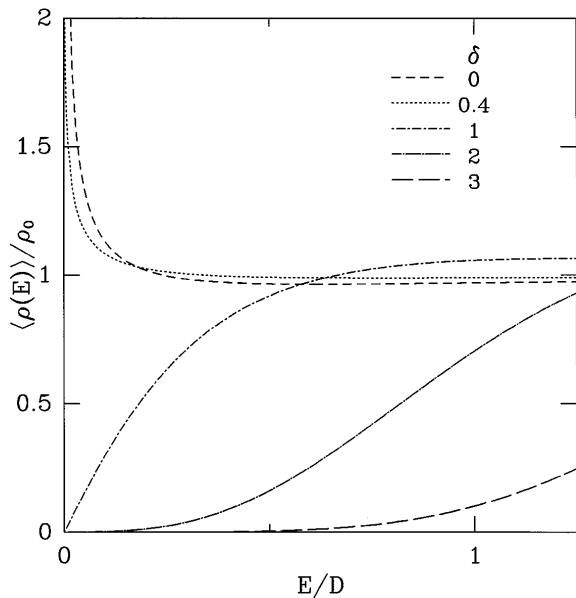


FIG. 1. Energy dependence of the disorder-averaged density of states for the commensurate case for various values of the dimensionless parameter δ [see Eq. (5)], which is a measure of the deviation from criticality. The density of states is singular at the Fermi energy ($E = 0$) when $\delta < \frac{1}{2}$. This parameter range corresponds to a Griffiths phase. Note that only far from criticality ($\delta \gg 1$) is there effectively a gap in the system. This contrasts with the disorder-free case, for which there is always a gap except at the critical point.

Dynamical critical exponent z .—This relates the scaling of energy (or time) scales to length scales. A crude scaling argument [25] implies that $\langle \rho(E) \rangle \sim E^{1/z-1}$. Thus for the commensurate case, to leading order in δ , $z = 1/2\delta$, in agreement with Fisher [3] and Young and Rieger [5]. This is a particularly striking result because it shows that (i) z is not universal, and (ii) z diverges at the critical point. The latter implies logarithmic scaling and activated dynamics [2]. In contrast for the incommensurate case, the density of states is finite and constant at low energies and so $z = 1$, as in the absence of disorder.

Finite size scaling.—Monthus *et al.* [26] studied an equation equivalent to (3) with $V(x)$ real and $\Delta = 0$ [23]. They have shown that on a line of length L , for a typical potential $V(x)$ the lowest eigenvalue E_0 scales like $E_0^2 \sim \exp(-cL^{1/2})$, where c is a constant. This is consistent with the scaling of $\ln E_0$ with $L^{1/2}$ at the critical point found numerically [5]. The average $\langle E_0^2 \rangle \sim \exp(-dL^{1/3})$ where d is a constant [26], showing the discrepancy between *average* and *typical* values.

Correlation lengths.—Fisher [3] stressed the distinction between average and typical correlations. If $C_{ij} \equiv \langle A_i A_j \rangle$ denotes a correlation function of a variable A_i then the average correlation function $C_{av}(r) \equiv \frac{1}{L} \sum_{i=1}^L C_{i,i+r}$ is what is measured experimentally. Away from the critical point $C_{av}(r) \sim \exp(-r/\xi_{av})$, where ξ_{av} is the average correlation length. However, $C_{av}(r)$ is dominated by rare pairs of spins with $C_{ij} \sim 1$. In contrast, with probability one $C_{i,i+r} \sim \exp(-r/\xi_{typ})$ where ξ_{typ} denotes the typical correlation length. It is distinctly different from ξ_{av} ($\xi_{typ} \ll \xi_{av}$), having a different critical exponent.

The localization length is useful because it is proportional to the typical correlation length for quantities that are diagonal in the fermion representation [27]. Consequently, the results in Table II imply that for the commensurate case the typical correlation length $\xi_{\text{typ}} \sim \lambda(0)^{-1} \sim \Delta^{-1}$ is consistent with previous work [3,28]. This critical exponent is not modified by the presence of disorder. In contrast, for the incommensurate case, $\lambda(0)$ is *finite* at the critical point.

Distribution functions.—Both Fisher [3] and Young and Rieger [5] considered the distribution functions for various quantities. The present approach can be used to obtain exact results by using known results for the distribution function for the density of states of one-dimensional conductors [29]. The cumulants (or irreducible moments) of the transverse susceptibility for the incommensurate case at criticality are

$$\langle (\chi_{zz})^n \rangle_c = \chi_0^n \frac{\Gamma(\frac{3}{2})\Gamma(n)^2}{\Gamma(n + \frac{1}{2})} \left(\frac{\pi D}{16T} \right)^{n-1}, \quad (7)$$

where χ_0 is the susceptibility in the absence of disorder. These moments completely determine the distribution function which has been extensively analyzed in Ref. [29]. For $T \gg D$ the distribution is Gaussian and centered on the mean. In contrast, for $T \ll D$ the distribution is extremely broad and asymmetric. The maximum of the distribution then occurs near $\chi_0 \exp(-1/s)/2s$, where $s = 16T/D\pi$, which is much less than the mean χ_0 . This shows the large discrepancy between typical and average values. Hopefully (7) can be generalized to the commensurate case and away from criticality.

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