Suppression of Chaotic Diffusion by Quenched Disorder

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It is shown that normal and anomalous chaotic diffusion can be totally suppressed by the presence of quenched disorder in the equations of motion. In special cases the problem can be mapped to random walks in random environments, where this effect is known as the Golosov phenomenon. [S0031-9007(96)01770-X]

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Chaos in dynamical systems is nowadays a well established mechanism for the generation of normal and anomalous diffusion processes. This holds for both Hamiltonian systems, such as the kicked rotator or the periodic Lorentz gas, and also for dissipative systems [1,2]. The study of normal and anomalous chaotic diffusion in the latter was initiated in [3–7] for various one-dimensional maps. Recently, within the advancement of periodic orbit theory, the thermodynamic formalism, and Levy flight statistics, these simple systems found renewed interest [8]. All these examples have in common the fact that their equations of motion are periodic in at least one variable. For the dissipative maps this was motivated by assuming that they capture essential aspects of the driven, damped motion of particles in periodic potentials [3–5]. Of course, periodicity of the static part of the underlying time-dependent potential is only one limiting case of the general situation. The other extreme of disordered potentials is much less studied in dynamical systems theory [9]. From many fields of physics, especially from solid state problems, it is known that static or quenched randomness may drastically alter macroscopic quantities such as transport coefficients. In the following we will report on such an effect for dynamical systems, namely, the total suppression of normal or anomalous chaotic diffusion by quenched randomness in the equations of motion. This is a nontrivial effect, since the mean-square displacement will remain finite, although chaotic transport is *not* inhibited locally.

We will concentrate on one-dimensional maps of the type studied in [3–8]. They have the general form $x_{t+1} = f(x_t) = x_t + F(x_t)$, with $F(x)$ periodic in x. The periodicity interval, which we set equal to one, i.e., $F(x) = F(x+1)$, defines cells or half-open intervals $A_i = [i, i+1), i \in \mathbb{Z}$, on the real axis. We will modify these dynamical systems by randomly changing $F(x)$ in each cell A_i to a function $F^{(i)}(x)$ resulting in

$$x_{t+1} = x_t + F^{(i)}(x_t) \tag{1}$$

for $x_t \in A_i$. This corresponds to a spatially random variation of the driving force felt by the particle. A natural choice for $F^{(i)}(x)$ consists of random shifts of F,

$$F^{(i)}(x) = F(x) + \varepsilon(i)\,. \tag{2}$$

In order to avoid complications connected with a global bias we assume, as usual, the symmetry $F(-x) = -F(x)$, and further that the $\varepsilon(i)$ are independent, identically distributed random variables with a symmetric distribution function $p(\varepsilon) = p(-\varepsilon)$, implying $\overline{\varepsilon(i)\varepsilon(j)} \propto \delta_{ij}$ and $\overline{\varepsilon(i)} = 0$. Through the cell index i, defined as $i = [x]$, the largest integer smaller than x, the term $\varepsilon(i)$ is recognized as a piecewise constant random function of x. In contrast to previous studies [3], where time-dependent noise was added to the deterministic dynamics, the random term $\varepsilon(i)$ remains constant in time; Eq. (1) is still deterministic, it describes a dynamical system with *quenched* randomness.

Let us now investigate the effect of this static randomness first for the simplest maps which, in the absence of disorder $[\varepsilon(i) = 0]$, exhibit chaotic diffusion. These are systems where $F(x)$ varies linearly in each cell, i.e., $F(x) = a\{x\} - a/2$ with $\{x\} = x - [x]$. Since the slope of $f(x)$ is $a + 1$, these maps are chaotic for $a > 0$ and show chaotic diffusion for $a > 1$. The dashed graph in Fig. 1(a) is an example with $a = 3$. The diffusive motion for this ordered case is verified by the linear increase of the mean-square displacement $\sigma^2(t) = \langle x_t^2 \rangle - \langle x_t \rangle^2 = 2Dt$ with the correct diffusion constant $D = 1/4$ [4] [dashed line in Fig. 1(b)]. This and the following results for $\sigma^2(t)$ were obtained numerically by iterating ensembles of 2×10^4 points (initially distributed homogeneously or inhomogeneously in one cell) for 10^6 (occasionally, 10^7) time steps. An example of a map with binary disorder, $\varepsilon(i) = \pm 1/2$ in Eq. (2), is shown as a full line in Fig. 1(a). Now, with disorder $\varepsilon(i) \neq 0$, a very different behavior is observed: $\sigma^2(t)$ saturates and remains bounded for large times. As is seen from Fig. 1(b) this is true for discrete random variations as well as for continuously distributed random variables $\varepsilon(i)$. We emphasize that for both cases there exists no obvious reason why the spreading of the distribution $\rho_t(x)$ should be limited, because the *a priori* probability for reaching one of the neighboring cells is always finite. More explicitly, *independent* of the chosen sequence $\varepsilon(i)$, a fraction $p = 1/4$ of a homogeneous distribution in some cell A_i is always transferred to the right neighboring cell A_{i+1}, the same fraction to

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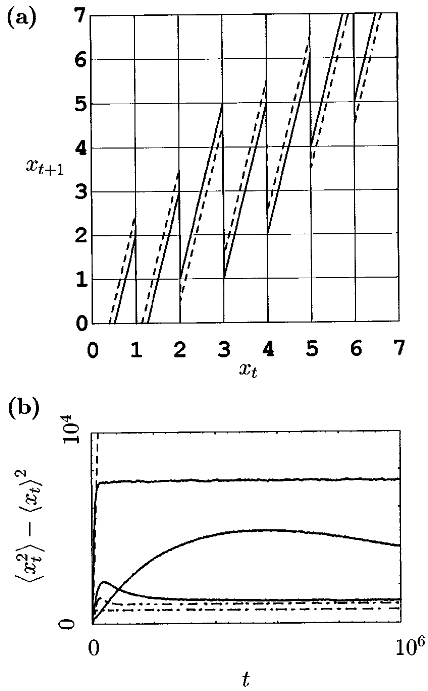

FIG. 1. (a) Simple piecewise linear maps corresponding to a periodic (dashed) and a random driving force (bold). The mean-square displacement (b) increases linearly for the former (dashed line) and saturates in the latter case as shown for several disorder realizations [full lines: $\sigma^2(t)$ for $\varepsilon(i) = \pm 1/2$; dot-dashed graphs: $\varepsilon(i)$ equally distributed in $(-1/2, +1/2)$]. There exist environments where a first constant level is observed only after more than $t = 10^6$ iterations [$(0.5-1.0) \times 10^7$ for the second graph from the top].

the left cell A_{i-1}, and one quarter remains within the cell A_i. From this point of view there is no difference between the homogeneous situation $[\varepsilon(i) = 0]$ and in the inhomogeneous case $[\varepsilon(i) \neq 0]$. The randomness affects only the last quarter, which is mapped into one or both of the next-nearest cells $A_{i\pm 2}$. Note also that the degree of chaoticity as measured by the Lyapunov exponent (Fig. 1: $\lambda = \ln 4$) is not altered by the random shifts.

The explanation of this localization effect in the case of quenched randomness follows from the following connection. For the map $f(x)$ with discrete random shifts as in Fig. 1, the cells A_i define a (generating) Markov partition [10]. This implies that the evolution of piecewise constant distributions $\rho_t(x)$ (constant in the cells A_i) is fully equivalent to a Markov process, i.e., the content $\pi_i(t) = \int_i^{i+1} \rho_t(x)dx$ of cell A_i at time t is iterated according to

$$\pi_j(t+1) = \sum_i \pi_i(t) p_{ij} \,. \tag{3}$$

For the above piecewise linear map with $\varepsilon(i) = \pm 1/2$ (Fig. 1) the only nonzero transition probabilities p_{ij} are given by $p_{ii} = p_{i,i\pm 1} = 1/4$ and $p_{i,i\pm 2} = [1/2 \pm \varepsilon(i)]/4$ [11]. Such a model defines a discrete random walk in a locally asymmetric random environment. The above localization effect, i.e., $\sigma^2(t)$ remaining finite for $t \to \infty$, is known as the *Golosov phenomenon* in the random walk literature [2]. Inspired by Sinai's work [12], it was proven rigorously for systems with only nearest-neighbor transitions by Golosov [13]. Reversing the above arguments which led us from iterated maps to random walks, it is obvious that also for the latter systems there exist realizations in terms of dynamical systems. These consist of piecewise linear chaotic maps of the form of Eq. (1), with a typical example shown in Fig. 2. Again, the cells A_i provide a Markov partition for this system. The segments of length p_{ii} and $p_{i,i\pm 1}$ in each unit cell, where the map $f(x)$ is linear, correspond to the nonzero transition probabilities p_{ii} and $p_{i,i\pm 1}$ of the associated Markov chain [14]. So far we have seen that the dynamical systems defined in Figs. 1 and 2 can both be mapped to random walk models with (locally) asymmetric random transition probabilities to next-nearest and nearest neighbors, respectively. Because of the short ranged correlations in the quenched disorder, they belong to the same universality class [2].

An intuitive picture for the relevant physical processes is obtained from the continuum limit of these discrete random walk models, which is the Brownian motion in a spatially random force field $\tilde{F}(x)$ [2]. In this limit the dynamics is governed by the Langevin equation

$$\dot{x}(t) = -\frac{\partial \tilde{V}}{\partial x}[x(t)] + \xi(t) \tag{4}$$

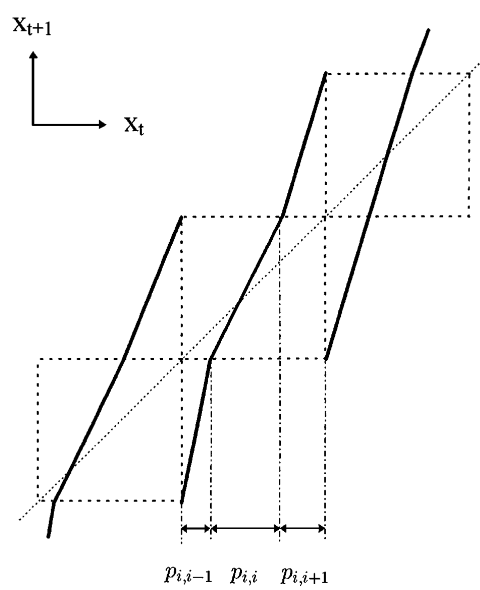

FIG. 2. An example from the class of iterated maps for which the asymptotically finite mean-square displacement follows rigorously from the work of Sinai and Golosov [12–14]. Also shown by dashed lines are the unit squares of the integer grid [see Fig. 1(a)] along the bisectrix. The indicated intervals p_{ii} and $p_{i,i\pm 1}$ mediate the transitions from the ith cell to itself and its neighbors, respectively.

with Gaussian white noise $\xi(t)$. The important point is that the associated potential $\tilde{V}(x) = -\int^x \tilde{F}(x')dx'$ itself can be thought of as a spatial realization of a Brownian path. The resulting statistical self-similarity of the potential $\tilde{V}(Lx) \simeq L^{1/2}\tilde{V}(x)$ implies the occurrence of deeper and deeper potential wells as the particle proceeds. The work of Sinai and Golosov shows that an ensemble of initially close particles moves in a coherent fashion from one deep minimum to the next deeper potential well [15]. In this stepwise process it is typically one minimum which dominates and therefore determines the (finite) width $\sigma^2(t)$ of the ensemble. Since the random environment in the neighborhood of these minima is the same only in a statistical sense, one still observes, for a fixed environment, fluctuations in $\sigma^2(t)$. These fluctuations become extremely rare for large times t as follows from an Arrhenius argument [2] which states that the typical time to overcome the ever increasing relevant potential barriers increases exponentially with the barrier height.

Applying this picture of a thermally activated process in a random Brownian landscape to dynamical systems presupposes the existence of a Markov partition. The results of Fig. 1 for continuous distributions of shifts $\varepsilon(i)$, however, show that the observed localization phenomenon is not bound to the existence of a Markov partition. An alternative, direct connection to random walk landscapes is obtained by rewriting the evolution equation (1) as

$$x_{t+1} - x_t = -\frac{\partial V}{\partial x}(x_t)\,, \tag{5}$$

which can be regarded as a discrete version of a gradient descent algorithm (operating in an unstable regime if the system is chaotic). The associated "potential" $V(x)$, although different from $\tilde{V}$ of Eq. (4), varies again like a random walk trajectory. This follows, for maps of the form (1) with continuous or discrete shifts as in Eq. (2), from the fact that the increments $\Delta V(n) = V(n+1) - V(n)$, $n \in \mathbb{Z}$, are independent random variables which are simply given by $\Delta V(n) = -\varepsilon(n)$, because $\int_0^1 F(x)\,dx = 0$ due to the symmetry of F. Similarly, for the system of Fig. 2, $V(x)$ is piecewise parabolic with random increments $\Delta V(n) = p_{n,n-1} - p_{n,n+1}$ [16]. We observed in all our numerical simulations that the distribution $\rho_t(x)$ was concentrated near a local minimum of the associated $V(x)$ during the quasistationary episodes of the evolution (where mean value and variance remain approximately constant). Thus the potential $V(x)$ of Eq. (5), together with the intrinsic stochasticity of the initial conditions (deterministic chaos), appears to play the same role as $\tilde{V}(x)$ of Eq. (4) in connection with the external noise $\xi(t)$.

Obviously, the random walk behavior of $V(x)$ is not connected to a piecewise linear variation of the maps, and one also expects the localization phenomenon for more general nonlinear maps. In order to test this conjecture let us consider cases which, in the absence of disorder $[\varepsilon(i) = 0]$, lead not only to diffusive but to superdiffusive motion. Such a behavior is obtained, e.g., for the choice $F(x) = -\cos(\pi\{x\})$ in Eqs. (1) and (2). For $\varepsilon(i) = 0$ this map belongs to the classes studied in [7]: The reduced map $f_r(x) = \{f(x)\}$ exhibits marginally stable fixed points at the cell boundaries which lead to intermittent enhanced diffusion with $\sigma^2(t) \propto t^2$ [case $z = 2$ in [7]]. Naively, one would expect that disorder, i.e., $\varepsilon(i) \neq 0$, results in normal diffusion with $\sigma^2(t) \propto t$ because the marginally stable fixed points of $f_r(x)$ become unstable or vanish as a result of the perturbations $\varepsilon(i)$. The numerical investigations, however, show again that the quenched disorder results in a strong localization of the trajectories (Fig. 3).

Obviously, there are many modifications of the periodic case $F(x) = F(x+1)$ where the localization effect can be expected to occur. The mechanism responsible for the trapping of the trajectories consists of the diverging fluctuations of the random walk potential $\tilde{V}(x)$ in Eq. (4) or $V(x)$ in Eq. (5) [17]. In Eq. (1), with (2), it is the random bias in the "force" $F^{(i)}(x)$ which leads to the random walk in $V(x)$. The same effect is obtained for random phase shifts of $F(x)$ in each cell, since the increments

$$V(n+1) - V(n) = -\int_n^{n+1} F(x + \varphi(x))dx$$

are also random for a random function $\varphi(x)$. The example in Fig. 2 can be regarded as a combination of a random bias with a random phase in the force field.

We have restricted ourselves to everywhere expanding maps (Figs. 1 and 2) and systems with nonhyperbolic behavior at isolated points (Fig. 3), respectively. An interesting and well studied class in the context of chaotic

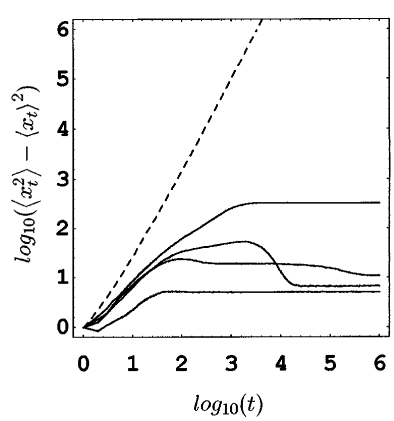

FIG. 3. Plots of the mean-square displacement on a doubly logarithmic scale for maps of the form $x_{t+1} = x_t - \cos(\pi\{x_t\}) + \varepsilon([x_t])$. The asymptotically linear behavior of the dashed graph $[\varepsilon(i) = 0)]$ reproduces the result $\sigma^2(t) \propto t^2$ [7]. The remaining graphs are obtained for several realizations of random sequences $\varepsilon(i)$ with the $\varepsilon(i)$ independently and equally distributed in the interval $(-1/2, +1/2)$: After $10^3 - 10^4$ iterations, all 2×10^4 initial points are effectively trapped in a local minimum of the potential $V(x)$ of Eq. (5).

diffusion are continuous maps such as the climbing sine map with $F(x) \propto \sin(2\pi x)$ [3–5]. Because of their massively nonhyperbolic behavior, these systems already exhibit a complex dependence on their parameters in the absence of disorder. One finds various transition scenarios between chaotic diffusion, localized chaotic or periodic motion (attractors), etc. [4,5]. Modifying such maps into random functions leads again to the suppression of chaotic diffusion. Since, however, the nonhyperbolic regions typically survive the introduction of disorder, the other above mentioned localization mechanisms persist locally. As a result, one gets dynamical systems where the different localization mechanisms interact or compete in a nontrivial manner. In this context it is worth discussing the effect of thermal noise on the localization phenomenon. The addition of time-dependent noise in the deterministic equation (5) will obviously result in a system intermediate between that of Eq. (5) and the system described by the stochastic differential equation (4). In any case, it does not destroy the random walk structure of the potential. Consequently, the presented generalization of the Golosov phenomenon to deterministic dynamical systems appears to be very robust against noise, and is probably the dominant localization mechanism in noisy disordered, nonhyperbolic systems. A similar discussion holds if $F(x)$ is chosen as a continuous random function at the outset without reference to any periodic behavior. In any case, such disordered maps deserve further investigations.

In summary we have shown that disorder in dynamical systems can lead to a total suppression of normal and anomalous chaotic diffusion. This effect can be regarded as a generalization of the Golosov phenomenon known from random walks in random environments to a large class of dynamical systems. We introduced models which show that this localization effect can occur without change of the chaotic properties of the system. We presented numerical results and identified dynamical systems where rigorous results apply.

The numerical computations were performed on the Cray Y-MP computers of the Rechenzentrum der Universität Kiel. Support from the latter is gratefully acknowledged.

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[15] The mean displacement is known to grow anomalously as $\langle x(t)\rangle = \xi(t) \ln^2 t$ with $\xi(t)$ a random function of $O(1)$ [2,12,13].

[16] One often associates with this random environment an "effective" discrete potential $\Phi(n)$ with random increments $\Delta\Phi(n) = \ln(p_{n,n-1}/p_{n,n+1})$. In the continuum limit this becomes the potential $\tilde{V}$ of Eq. (4). For $p_{n,n} = p_0 =$ const the increments $\Delta V(n)$ and $\Delta\Phi(n)$ are related by $\Delta V(n) = (1 - p_0)\tanh[\Delta\Phi(n)/2]$, implying $\overline{[\Delta\Phi(n)]^2} \propto \overline{[\Delta V(n)]^2}$ for $|\Delta\Phi(n)| \ll 1$. From the rigorous treatments of this system we know that $\sigma(t \gg 1)$ scales as $\{\overline{[\Delta\Phi(n)]^2}\}^{-1}$ in the small disorder limit [2,13]. By simple scaling arguments this should hold more generally with $\Delta\Phi$ replaced by ΔV.

[17] An unbounded $V(x)$ in Eq. (5) can also occur for the damped motion of *driven* particles in bounded static potentials $\hat{V}(x)$, because, via stroboscopic maps, $V(x)$ is rather related to the typically unbounded time-dependent potential [e.g., $\hat{V}(x,t) = \hat{V}(x) + \alpha x \cos(\omega t)$].