Sustaining Chaos by Using Basin Boundary Saddles

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(Received 8 July 1996)

We present a general method for preserving chaos in nonchaotic parameter regions by using the natural dynamics of the system, and apply it to a CO_2-laser model. Chaos is preserved by redirecting the flow towards the chaotic region along unstable manifolds of basin boundary saddles, with the use of small infrequent parameter perturbations. [S0031-9007(96)01784-X]

PACS numbers: 05.45.+b

Although numerous areas in science are now known to exhibit chaos as a natural occurrence, many areas would benefit from the inducement of chaos. In biology, the disappearance of chaos may signal pathological phenomena [1,2]. In mechanics, chaos could be induced in order to prevent resonance, as is the case for a system of coupled pendulums presented in Ref. [3], where one can excite chaotic motion of several modes spreading the energy over a wide frequency range [4]. In optics [5], material damage is caused by lasers having a peak intensity at a given temporal frequency, so chaos is desirable since it has broadband spectra. It has also been suggested that chaos occurs for normal machine tool cutting, making chaos preservation a desired control for deeper than normal cutting [6].

In Ref. [2] a method has been proposed to maintain chaos in a regime where only chaotic transients exist. The method was implemented based on accurate analytical knowledge of the dynamical system, and requires *a priori* phase-space knowledge of escape regions from chaos, as well as preiterates of these regions. Another analytic scheme for sustaining chaos was introduced in Ref. [7] that uses state variable control. In the spirit of the OGY method of control [8], the technique presented here makes use of quantities which are measurable experimentally and thus can be applied to real data taken directly by time series. In particular, we show how to sustain chaos by using the dynamics of a governing saddle. An alternative method which also maintains chaos using time series [9] requires monitoring and adjusting the system prior to entering an escape region of the attractor by several iterates. In contrast our method requires only examining behavior near a saddle.

The situation we address occurs when there are chaotic transients in the presence of another nonchaotic attractor. The method maintains chaos by using the natural dynamics of unstable states lying on the basin boundary separating a periodic attractor from chaotic transients, which we call basin saddles. (Technically, there is only one attractor since chaos is a transient. However, there is still a stable manifold which separates the chaotic transient from the periodic attractor [10].) Once the flow enters a neighborhood of a basin saddle, we use small perturbations of an accessible system parameter to redirect the flow towards the chaotic transient region. This is done by a targeting technique which uses the linearization of the flow about the saddle. A probability distribution of escape times is used to optimize chaos preservation by targeting regions having long chaotic transients, thus minimizing the number of parameter fluctuations.

In what follows we understand by "escape region" a region such that if an iterate enters the region, subsequent iterates are rapidly drawn to an attractor. (The same notion is called "loss region" in [2] and [9].) Instead of preventing escape to an attractor in advance, our approach is to let the system enter a region containing a basin saddle and then redirect the flow back into the chaotic region, by using a targeting technique which uses the natural dynamics around the saddle. This makes the parameter changes very infrequent and, in fact, minimal.

The model.—The targeting method for maintaining chaos is applied to a periodically driven CO_2 laser,

$$\frac{du}{dt} = -u[\delta \cos(\Omega t + \phi) - z],$$
$$\frac{dz}{dt} = -\epsilon_1 z - u - \epsilon_2 z u + 1\,, \tag{1}$$

where u and z denote (scaled) intensity and population inversion. The original model was introduced in Ref. [11], and control and tracking of unstable orbits are done in Ref. [12]. The control parameter δ represents the amplitude of the drive, and ϵ_1, ϵ_2, and ϕ are fixed.

At $\delta_c \cong 1.84$ a crisis occurs between a period-two saddle and a chaotic attractor. Just prior to δ_c, there exist two attractors, one periodic and one chaotic. The stable manifold of a period-two saddle lying on the basin boundary splits the local phase space about the saddle in the following manner: orbits having initial conditions lying to the right, say, and near one saddle converge

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to the chaotic attractor [10]. Orbits starting to the left converge to the period-two attractor. For δ slightly past δ_c, a horseshoe is created from the right; i.e., the unstable manifold to the right of the stable manifold crosses the stable manifold. [See Fig. 1(a) near S_a.] Almost all points in the region near the saddle now converge to a period-four orbit which has period doubled off the period-two branch. Figure 1(a) shows the manifolds of phase space of the defining period-two basin saddle, labeled by S_a and S_b. Figure 1(b) shows a chaotic transient settling into a period-four attractor after the iterate enters the escape region to the left of the upper saddle. (Every other iterate is shown.)

For our method of maintaining and optimizing the length of chaotic transients, we need to identify regions near the basin saddles which contain points of long chaotic transients. The method will redirect the flow towards these regions once it crosses the basin boundary of the attractor. We consider a distribution of escape times for trajectories starting near S_a, close to the unstable manifold of S_a and to the right (i.e., the side with chaotic transients). These points lie on a line segment, L_a near S_a.

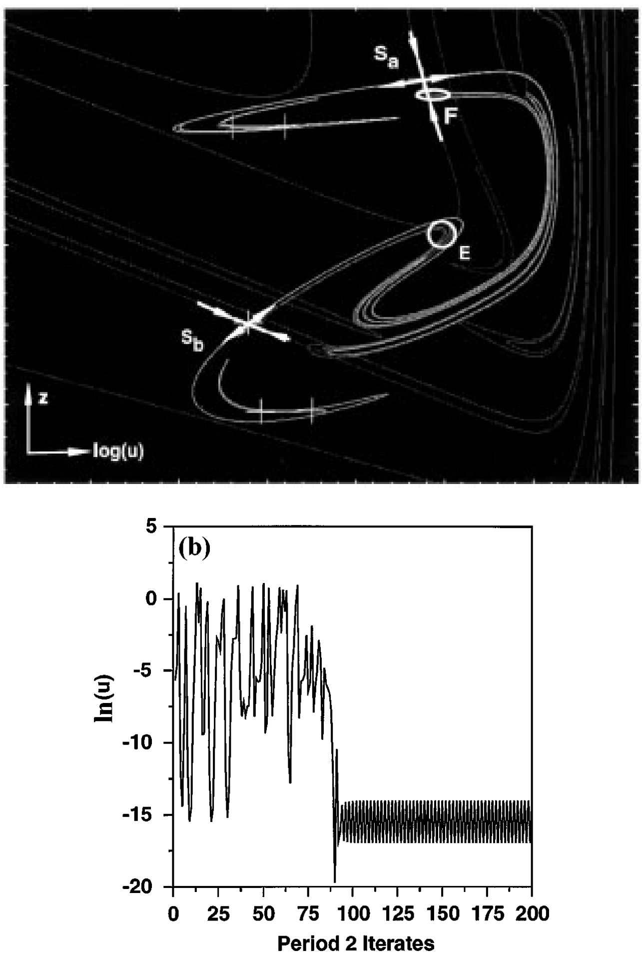

FIG. 1. (a) Stable (grey) and unstable (white) manifolds associated with basin boundary period-two saddle (S_a, S_b) at $\delta = 1.88$. The crosses represent the period-four attractor. The chaotic transients lie near the unstable manifold. (b) Time series for $\ln(u)$ at $\delta = 1.88$ illustrating a chaotic transient landing on a period-four attractor (every other iterate is shown).

In Fig. 2, the horizontal axis represents equally spaced intervals on L_a, near S_a. For the points in such an interval we calculate the fraction of points having escape times in time intervals represented on the vertical axis. In other words, given a time interval on the vertical axis and a space interval on the horizontal axis, the intensity of the grey scale indicates the fraction of points in that space interval having escape times in the chosen time interval. From this figure we see that there is a wide distribution of points near the saddle which escape very quickly to the attractor as well as points with very long escape times. If one computes the fraction of points having a given mean escape time, one finds an exponential distribution, in agreement with results reported in [13]. Moreover, since preimages of points in the escape regions essentially cover the chaotic transient region, applying perturbations to prevent the flow from entering the escape regions, as done in [2] and in [9], may kick the dynamics into faster escaping regions.

In what follows we write (1) in generic form as $\boldsymbol{\phi}' = F(t, \boldsymbol{\phi})$. A Poincare map of this flow is obtained by sampling the system with the frequency of the drive, giving

$$\mathbf{x}_{n+1} = T(\mathbf{x}_n, \delta) = \boldsymbol{\phi}(1, 0; \mathbf{x}_n, \delta), \tag{2}$$

where δ is the parameter we adjust to maintain chaos. The Jacobian of the map T, calculated at the unstable orbit, is given by $A = (\partial \phi / \partial \mathbf{x}_n)(1, 0, \mathbf{x}_n, \delta)|_{\mathbf{x}_n = \mathbf{S}_a}$.

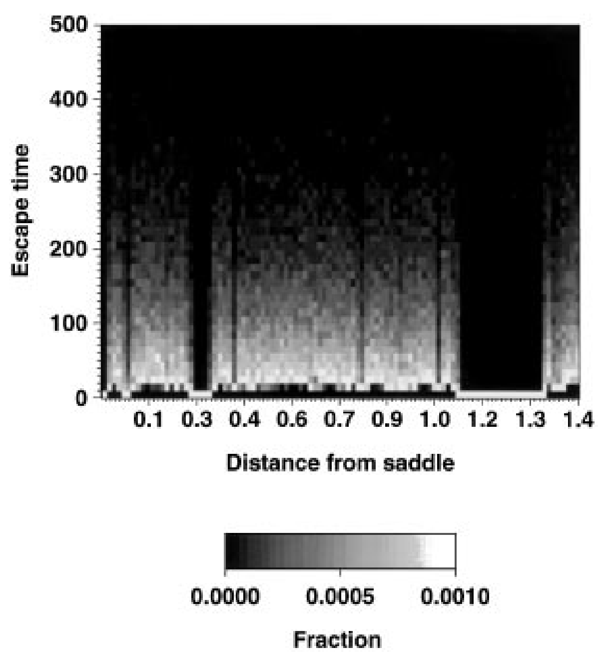

FIG. 2. Probability distribution of escape times as a function of distance from the saddle S_a for points inside the chaotic region to the right of S_a near the unstable manifold (see text for details).

The algorithm.—The algorithm we propose perturbs δ once an iterate of T^2 enters a neighborhood of the period two basin boundary basin saddle (S_a, S_b), where $S_a = T^2(S_a)$, and $S_b = T(S_a)$. For our technique, we linearize T^2 about both S_a and S_b. Based on this linearization we derive a formula for the parameter perturbation that can be used to regenerate chaos. The same procedure has to be applied for both S_a and S_b due to the fact that the chaotic transients visit neighborhoods of both S_a and S_b (which contain escape regions), even when the map we consider is T^2.

In order to make use of basin saddle information, we restrict ourselves to a local region, D_{loc}, of phase space near a saddle, say S_a. We call the region to the left of S_a contained in D_{loc}, the basin escape region. (A similar description holds for a local region near S_b.)

The stable manifold of the period-two saddles separates the region with transient chaos from the period-four attractor. We wish to send the iterates, once they enter the basin escape region, back to the opposite side of the stable manifold, preferably close to the unstable manifold of the saddle where the natural dynamics will send the iterates farther into the chaotic region. By restricting the action of the dynamics to D_{loc}, we may examine the perturbations acting on a linearized system.

To derive the formula for the parameter perturbations let $\boldsymbol{\xi}_n = \mathbf{x}_n - \mathbf{S}_a$ and consider the map $\boldsymbol{\xi}_{n+1} = \mathbf{P}(\boldsymbol{\xi}_n, \delta) = \mathbf{T}^2(\mathbf{x}_n, \delta) - \mathbf{S}_a(\delta)$, which has fixed point $\boldsymbol{\xi}_F = 0$. We linearize this map about its fixed point, and using the same notation as in [8], we obtain

$$\boldsymbol{\xi}_{n+1}\delta_n \mathbf{g} + [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot (\boldsymbol{\xi}_n - \delta_n \mathbf{g}), \qquad (3)$$

where A has been expressed in terms of left and right eigenvectors, and $\mathbf{g} \equiv [\partial \mathbf{S}_a(\delta)/\partial\delta]_{\delta=\delta_0} \cong 1/(\bar{\delta} - \delta_0)\boldsymbol{\xi}_F(\bar{\delta})$, for some $\bar{\delta}$ close to the operational parameter δ_0.

We target a desired region near the saddle on the side with chaos, based on the distribution in Fig. 2, where we know there is a high probability of obtaining long chaotic transients. In (3), we require that $\boldsymbol{\xi}_{\mathrm{tar}} = \mathbf{x}_{\mathrm{tar}} - \mathbf{S}_a$, where $\mathbf{x}_{\mathrm{tar}} \epsilon \mathbf{L}_a \cap \mathbf{D}_{\mathrm{loc}}$ is a chosen target point. We choose x_{tar} such that it is contained in a region where long chaotic transients emerge before the flow escapes again. Equation (3) becomes

$$\boldsymbol{\xi}_{\mathrm{tar}} = \mathbf{x}_{\mathrm{tar}} - \mathbf{x}_F \cong [\lambda_u \mathbf{e}_u \mathbf{f}_u + \lambda_s \mathbf{e}_s \mathbf{f}_s] \cdot (\boldsymbol{\xi}_n - \delta_n \mathbf{g}). \qquad (4)$$

Multiplying through in (4) by $\mathbf{f}_s$,

$$\delta_n \equiv \frac{(\lambda_s \boldsymbol{\xi}_n - \boldsymbol{\xi}_{\mathrm{tar}}) \cdot \mathbf{f}_s}{(\lambda_s - 1)\mathbf{g} \cdot \mathbf{f}_s}$$

$$\equiv \mathbf{C}_1 \cdot (\lambda_s \boldsymbol{\xi}_n - \boldsymbol{\xi}_{\mathrm{tar}}). \qquad (5)$$

We remark that applying small changes in parameters given by (5) may lead to large changes in preimages of escape regions, but the saddle location varies only slightly, implying that the method is robust with respect to perturbations.

Numerical examples.—In the first example, the sustained chaotic time series for $\ln u$ is shown in Fig. 3 along with the corresponding parameter perturbations. Notice that only five perturbations to δ were performed to maintain 500 iterates of T^2, i.e., chaos is sustained for 1000 iterates of T. The noise level was 1%, and the parameter perturbations occurring in this case are below 0.6. We targeted certain regions near the unstable manifold which generate long chaotic transients. This is efficient since four of the five perturbations necessary to maintain chaos were larger than the mean chaotic transient time, which was found to be approximately 27 iterates.

One of the main assumptions in our method is that just before entering the basin escape region the iterates must pass near either saddle S_a or S_b. We show that iterates contained in a region not local to the saddles, but nevertheless an escape region, can be directed toward D_{loc} which contains a basin saddle. We pick a region inside the chaotic transients which lies in the basin of the period-four attracting orbit, and label this region, region E, in Fig. 1(a).

The idea is to redirect the iterates from such a region by targeting a certain region or point inside the chaotic regime by using, for example, the algorithm introduced in [14]. For our flow we have found that the region near the upper saddle to the right (let us call it region F) is accessible from region E, in only one iterate by small amplitude perturbations of the parameter. The reason is that the image under the flow of a point in E lies inside F. If this was not the case we would have to target several intermediate points, as shown in [14], before reaching region F. The phase-space sustained chaotic time series is shown in Fig. 4. The changes in the parameter used for targeting from region E to region F are 1 order of magnitude smaller than the other parameter changes and are not seen clearly in Fig. 4(b).

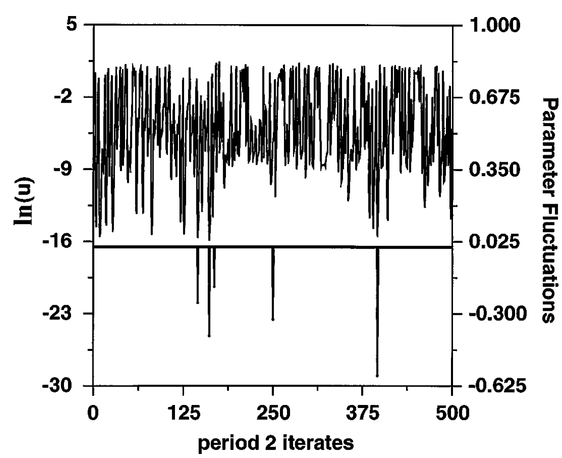

FIG. 3, Sustained chaotic time series for $\ln u$ and parameter fluctuations used to sustain chaos.

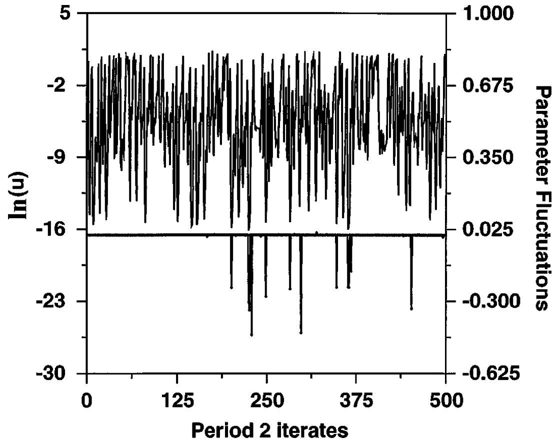

FIG. 4. Time series for $\ln u$ and corresponding parameter fluctuations used to sustain chaos based on monitoring region E in Fig. 1(a).

Finally, we have investigated several other escape regions such as region E, and we have noticed that they are mapped by the flow into one of the regions near the period-two saddle, so it is enough to apply parameter perturbations near the saddle.

IBS acknowledges support from the Office of Naval Research.

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