Gravitational Instantons and Minimal Surfaces

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(Received 3 October 1996)

We show that for every minimal surface in E^3 there is a gravitational instanton, an exact solution of the Einstein field equations with Euclidean signature and anti-self-dual curvature. The explicit metric establishing this correspondence is presented and a new class of exact solutions are obtained. [S0031-9007(96)01786-3]

PACS numbers: 04.20.Jb

Einstein field equations for anti-self-dual gravitational fields with Euclidean signature reduce to a complex elliptic Monge-Ampère equation [1]. On a complex manifold $\mathcal{M}$ of dimension 2, this equation is given by

$$(\partial\bar{\partial}u)^2 = \kappa\,{}^*1\,, \tag{1}$$

where u is the Kähler potential, ∂ denotes the holomorphic exterior derivative, the bar denotes complex conjugation, and ^{*1} is the volume element on $\mathcal{M}$. Here κ is a constant which determines the character of this equation, namely, it is elliptic for $\kappa > 0$ and hyperbolic if $\kappa < 0$. For $\kappa = 0$ we have the complex homogeneous Monge-Ampère equation which plays a prominent role in the theory of functions of two complex variables as it is the direct generalization of Laplace's equation [2]. The complex elliptic Monge-Ampère equation is the Einstein field equation governing instantons [3–9]. It is a formidable equation; however, some progress can be made by looking at its reduction in less than two complex variables. Well-known instanton solutions such as the Eguchi-Hanson [5] and Gibbons-Hawking [6] solutions can be obtained from a reduction of Eq. (1) in one real variable [7]. In this Letter we shall consider a reduction of the complex Monge-Ampère equation in two real variables. One possibility, which is rather naive but nevertheless very fruitful, is to simply assume that the Kähler potential depends only on the real parts of the two complex coordinates. For another reduction see [10]. So, if we lay down a local coordinate system on $\mathcal{M}$ with two complex coordinates $\zeta^1 = t + iz$, $\zeta^2 = x + iy$ and assume that $u = u(t,x)$ only, then we are led to the Kähler metric

$$\begin{aligned} ds^2 = {}& u_{tt}(dt^2 + dy^2) + u_{xx}(dx^2 + dz^2) \\ & + 2u_{tx}(dt\,dx + dy\,dz) \end{aligned} \tag{2}$$

and the Kähler 2-form

$$\begin{aligned} \omega = {}& u_{tt}\,dt \wedge dy + u_{xx}\,dx \wedge dz \\ & + u_{tx}(dx \wedge dy + dt \wedge dz) \end{aligned} \tag{3}$$

which is closed. Now from the reduction of the complex Monge-Ampère equation (1) we obtain

$$u_{tt}u_{xx} - {u_{tx}}^2 = \kappa \tag{4}$$

as the only Einstein field equation governing this class of anti-self-dual solutions. The two-dimensional real Monge-Ampère equation with constant right hand side (4) admits multi-Hamiltonian structure [11] which, by the theorem of Magri [12], provides proof of its complete integrability. Furthermore, Eq. (4) is equivalent to the equation for minimal surfaces in the elliptic case [13], whereas in the hyperbolic case it corresponds to the Born-Infeld equation [14]. Using this correspondence, we can write the metric for a class of instantons in the form

$$\begin{aligned} ds^2 = {}& \frac{\kappa + \phi_t^2}{\sqrt{1 + \kappa\phi_t^2 + \phi_x^2}}(dt^2 + dy^2) \\ & + \frac{1 + \phi_x^2}{\sqrt{1 + \kappa\phi_t^2 + \phi_x^2}}(dx^2 + dz^2) \\ & + 2\frac{\phi_t\phi_x}{\sqrt{1 + \kappa\phi_t^2 + \phi_x^2}}(dt\,dx + dy\,dz)\,, \end{aligned} \tag{5}$$

whereby the Einstein field equations reduce to the classical equation

$$(1 + \phi_x^2)\phi_{tt} - 2\phi_t\phi_x\phi_{tx} + (\kappa + \phi_t^2)\phi_{xx} = 0 \tag{6}$$

governing minimal surfaces if $\kappa = +1$, or the Born-Infeld equation for $\kappa = -1$. Hence we have the following:

Theorem.—For every minimal surface in E^3 the metric (5) provides an instanton solution of the Einstein field equations with Euclidean signature and anti-self-dual curvature.

The proof of this theorem consists of a straightforward check that for the metric (5) the only Einstein field equation is given by Eq. (6) governing minimal surfaces in E^3, and, provided that it is satisfied, the curvature 2-form is anti-self-dual.

Examples of instanton metrics that we can obtain through this correspondence are given by

$$\begin{aligned} ds^2 = {}& \frac{1}{r}[(dr + r\tan\theta\,d\theta)^2 + (dR + r\tan\theta\,d\phi)^2 \\ & + r^2(d\theta^2 + d\phi^2)] \end{aligned} \tag{7}$$

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and

$$ds^2 = \frac{1}{\sqrt{1 - \frac{a}{r^2}}}\Big[dr^2 + (r^2 - a)d\theta^2 + \Big(1 - \frac{a}{r^2}\cos^2\theta\Big)dy^2 + 2\,\frac{a\sin\theta\cos\theta}{r^2}\,dydz + \Big(1 - \frac{a}{r^2}\sin^2\theta\Big)dz^2\Big], \quad (8)$$

which is an asymptotically Euclidean metric that corresponds to the catenoid or the helicoid minimal surface, depending on

$$a = a_c^2, \quad a = -a_h^2, \quad (9)$$

respectively.

Finally, there is a remarkably simple instanton metric that contains an arbitrary analytic function

$$ds^2 = \xi(dt^2 + dx^2 + dy^2) + \frac{1}{\xi}(dz + \eta dy)^2, \quad (10)$$

where ξ and η are conjugate analytic functions satisfying

$$\xi_t = -\eta_x, \quad \eta_t = \xi_x, \quad (11)$$

the Cauchy-Riemann equations.

In the hyperbolic case we can use the correspondence between the Born-Infeld equation and the real hyperbolic Monge-Ampère equation in two dimensions [14] to write the metric in the form

$$ds^2 = V[(dx + Udt)^2 + (dz + Udy)^2] - \frac{1}{V}(dt^2 + dy^2), \quad (12)$$

where U, V satisfy the two-dimensional gas dynamics version of the Born-Infeld equation [15],

$$U_t = UU_x + V^{-3}V_x, \qquad V_t = (UV)_x, \quad (13)$$

as the Einstein field equations. These are the same as the Euler equations for Chaplygin gas with adiabatic index $\gamma = -1$. The multi-Hamiltonian structure of the Born-Infeld equation [15] is one of the richest of its kind, and its complete integrability is therefore well established. It can be solved by the classical hodograph method and from the hodograph solution of Eqs. (13), we arrive at the metric

$$ds^2 = (f - g)(dt^2 - dx^2 - dy^2) + \frac{1}{f - g}[dz + (f + g)dy]^2, \quad (14)$$

where $f(x + t)$ and $g(x - t)$ are two arbitrary functions. This solution is analogous to (10).

We have established that every solution of the equation for minimal surfaces in E^3 gives rise to an instanton solution of the Einstein field equations. This is a very large class of solutions which is impossible to explore fully in this Letter. We shall provide an exhaustive list in a forthcoming publication.

I thank Professor E. İnönü for his kind interest in this work.

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