Two-Dimensional Child-Langmuir Law

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By considering uniform emission of electrons over a finite strip of width W in a planar gap of gap separation D, we extend the classical one-dimensional Child-Langmuir law to two dimensions. The limiting current density in two dimensions $J_{\mathrm{CL}}(2)$ in units of the classical one-dimensional value $J_{\mathrm{CL}}(1)$ is found to be a monotonically decreasing function of W/D. More surprisingly, it is virtually independent of the external magnetic field that is imposed along the mean flow. These results were obtained from two different simulation codes, OOPIC and MAGIC. [S0031-9007(96)01721-8]

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The dynamics of charged particles in a gap has remained an area of considerable interest in vacuum microelectronics, crossed-field devices, and high power diodes. If too much space charge is injected into a gap, the resulting electric field becomes sufficiently high to reflect the injected particles, forming a virtual cathode. Laminar flow solution then ceases to exist and the flow in the gap shows oscillatory behavior. For a planar gap of gap separation D and gap voltage V, the maximum electron current density allowed for time-independent flow solution, $J_{\mathrm{CL}}(1)$ for the one-dimensional model, is given by the Child-Langmuir law [1–3],

$$J_{\mathrm{CL}}(1) = \frac{4\varepsilon_0}{9D^2}\left(\frac{2e}{m}\right)^{1/2} V^{3/2}, \tag{1}$$

where e and m are, respectively, the charge and mass of the emitted particle, and ε_0 is the free space permittivity. The value given in Eq. (1) is also called the limiting current density or critical current density. Implicit in Eq. (1) is the neglect of relativistic effects and the assumption of zero electron emission velocity.

While Eq. (1) is easy to derive, generalization to two dimensions is a formidable task analytically. Such a two-dimensional problem is of fundamental interest, for instance, for field emitters, where electrons are emitted only in the vicinity of the sharp tip. Rather than solving the entire problem for a field emitter, with the vast parameter space resulting from the complicated geometry of the emitter-gate-anode assembly, and from the dependence on the cathode material through the Fowler-Nordheim coefficients, we simply ask: How is Eq. (1) modified when electrons are allowed to emit only over a finite strip on a planar cathode? Investigation of this question then provides physical insight into virtual cathode formation when electrons are emitted only over a restricted region on the cathode, as in the case of a field emitter.

With the use of two-dimensional particle-in-cell (PIC) simulations, we analyze a planar gap with $D = 1$ cm and $V = 1$ kV. The anode-cathode plates are 8 cm in width, and the third dimension is infinite and uniform. Electrons are emitted from the cathode over a fixed strip of width W ($W < 8$ cm), centered around the midpoint. The emission current density J is uniform both in space and in time over that strip. Electrons are emitted with an initial energy of 0.1 eV.

Two PIC codes are used: MAGIC [4] and OOPIC [5]. The MAGIC code simulations were fully *electromagnetic*, with cell size of 2.5×10^{-4} m^2 and time step of 0.53 ps. The MAGIC simulations used a *periodic boundary condition* to close the computational area at the termination of the anode-cathode plates. The OOPIC simulations, in contrast, were *electrostatic*, with cell size of 8.0×10^{-4} m by 2.5×10^{-4} m and time step of 18.3 ps. A *nonreflecting boundary condition* was used in OOPIC to close the simulation. This differs from a periodic boundary in that particles are destroyed when they encounter the simulation edge, rather than being returned to the simulation on the other side. Both codes allow for the addition of a static magnetic field to simulate immersed flow (to be described below).

The simulations proceed by specifying the emission width W, and a low injected current density J. The current density J is then increased until oscillatory behavior is observed in the simulation, and this is taken to be the two-dimensional limiting current density $J_{\mathrm{CL}}(2)$ for that value of W. Note that this definition of limiting current density is consistent with that used to derive Eq. (1), namely, the absence of a time-independent solution for a given set of parameters, except that numerical simulation is used instead of analytic methods. The results are shown in Fig. 1, for both OOPIC and MAGIC simulations, for the case of zero external magnetic field. The good agreement between the two vastly different simulations, together with the recovery of the limit that $J_{\mathrm{CL}}(2)$ approaches $J_{\mathrm{CL}}(1)$ as $W \gg D$, strongly suggests validity of the results. The curves shown in Fig. 1 apply to all nonrelativistic voltages. This statement is verified by spot checks using different gap voltages and gap separations. A simple scaling of the governing equations (Poisson equation, the time-independent nonrelativistic force law and the time-independent continuity equation) shows that $J_{\mathrm{CL}}(2)/J_{\mathrm{CL}}(1)$ is a universal function of W/D for zero emission velocity.

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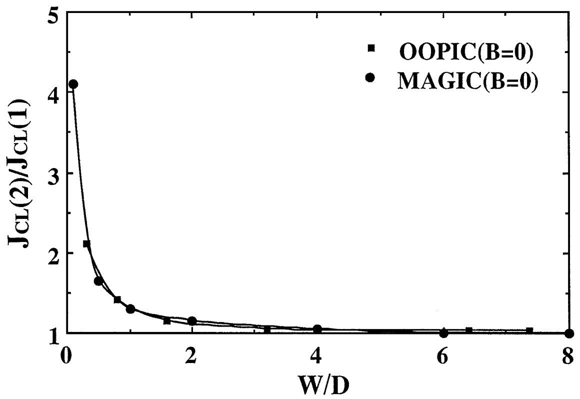

FIG. 1. The limiting current density in two dimensions, in units of the one-dimensional Child-Langmuir value, obtained from OOPIC and MAGIC for an unmagnetized gap.

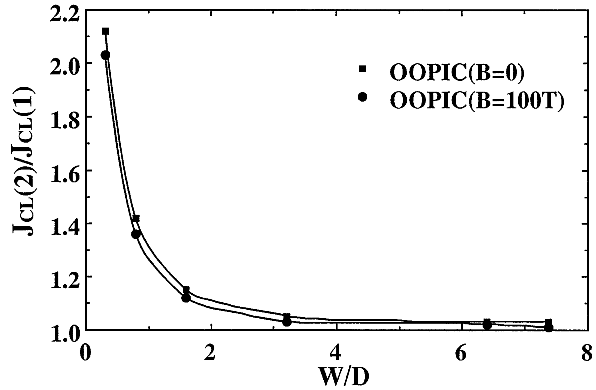

FIG. 2. The limiting current density in two dimensions, in units of the one-dimensional Child-Langmuir value, obtained from OOPIC for an unmagnetized $(B = 0)$ and a strongly magnetized $(B = 100\ \mathrm{T})$ gap. Similar results were obtained with MAGIC.

That universal function is exceedingly difficult to derive analytically. [See Eq. (2) below.]

The simulation shows the following scenario for the breakdown of the laminar flow as $J \rightarrow J_{\mathrm{CL}}(2)$. The electric field in the vicinity of the center of the emitting strip evolves progressively from the vacuum value to zero due to the space charge of the emitted electrons. At the point of virtual cathode formation, the electric field reverses direction, slowing and reflecting the emitted electrons. These electrons are reflected back into the cathode, which removes the space charge buildup, allowing the electric field to change sign and again pull electrons into the gap. The electric field at the center of the emitting strip oscillates from this point on, as the space charge repeatedly builds up to form a virtual cathode. This is similar to the description of one-dimensional virtual cathode oscillations [2], except that in two dimensions the virtual cathode always first appears at the center of the emitting region. Near the edges of the emitting strip, the electric field remains negative (i.e., points toward the cathode) at the transition from time-independent flow to turbulent flow.

The results described thus far correspond to a zero external magnetic field. We next consider the opposite limit, where a strong external magnetic field of 100 T parallel to the mean electron flow is imposed. The limiting current for this 2D "immersed flow" is calculated via simulation. The results are shown in Fig. 2, where the limiting current for the case of zero magnetic field is also included for comparison. In Fig. 2, we see that the two-dimensional limiting current is virtually independent of the external magnetic field. We have also simulated the intermediate magnetic field case of 0.01 T, at which the cyclotron frequency is on the order of the average plasma frequency; we find that the limiting current density is also almost identical to those shown in Fig. 2. Over the range simulated, $0.1 < W/D < 8$, the data may be fitted empirically by

$$\frac{J_{\mathrm{CL}}(2)}{J_{\mathrm{CL}}(1)} = +\frac{0.3145}{W/D} - \frac{0.0004}{(W/D)^2} \tag{2}$$

to within 5%, for all values of magnetic field: $B = 0, 0.01$, and 100 T.

The result that the magnetic field has only a very weak effect on the limiting current density is quite surprising, at first sight. Inspection of the phase space plots of the simulations, however, offers an explanation. The virtual cathode always forms near the center of the emitting strip in a sheath very close to the cathode. The space charge from the edges of the beam confines the transverse motion of the electrons in the sheath. For example, with $W/D = 0.32$ and $J = J_{\mathrm{CL}}(2)$, the maximum transverse energy of the electrons in the sheath is 1.7×10^{-4} eV for the unmagnetized gap compared with 2.8×10^{-12} eV for the strongly magnetized gap $(B = 100\ \mathrm{T})$. While there is 8 orders of magnitude difference in the two cases, the velocity of the electrons is too small for the magnetic field to produce a large force on the electrons in the sheath. (The maximum transverse energy outside of the sheath, however, is 7.3 eV for the unmagnetized gap compared with 1.1×10^{-11} eV for the strongly magnetized gap.) Thus, the transverse motion of the electron *in the vicinity of the virtual cathode* is negligible in both the zero magnetic field and the infinite magnetic field cases, for these simulations. The magnetic field does not, therefore, produce a large effect as shown in Fig. 2. The sheath physics is critical to the behavior of the entire gap [2,6,7]. In fact, the motion in the sheath dominates the formation of the virtual cathode, which explains the weak dependence of the critical current on magnetic field [8].

In summary, the classical Child-Langmuir law is generalized to two dimensions via particle simulation. The critical current density appears to be only weakly dependent on the axial magnetic field that confines the electron flow. The virtual cathode begins to form near the central region of the emitting surface, and very close to the surface.

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[8] For virtual cathode oscillators (vircators), it is well known that an applied magnetic field can have a large effect on the oscillatory behavior. See, e.g., *High-Power Microwave Sources*, edited by V. L. Granatstein and I. Alexeff (Artech House, Boston, 1987); B. V. Alyokhin *et al.*, IEEE Trans. Plasma Sci. **22**, 945 (1994). Vircators, however, are typically relativistic, with a sufficient current for self-magnetic field effects to become important. In addition, the virtual cathode is formed in drift spaces where the beam may have considerable transverse motion. Two-dimensional effects are then important, and, therefore, the applied magnetic field would have a strong effect.