

Analytic Scaling Solutions for Cosmic Domain Walls

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A relativistic generalization of a well-known method for approximating the dynamics of topological defects in condensed matter is constructed, and applied to the evolution of domain walls in a cosmological context. It is shown that there are self-similar “scaling” solutions, for which one can, in principle, calculate many quantities of interest without recourse to numerical simulations. Here, the area density in the scaling regime is calculated in various backgrounds. Remarkably good agreement with numerical simulations is obtained. [S0031-9007(96)01779-6]

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Topological defects formed at a cosmological phase transition are one of the two known ways of creating scale-free primordial density perturbations [1,2]. However, unlike rival inflation-based models [3], analytic calculations have not yet made much impact. There do exist analytic treatments of the defects and Goldstone modes arising from the spontaneous breaking of global symmetries [4,5], which are exact in the limit of large N , where N is the number of scalar fields, but they remain relatively undeveloped.

In this Letter a new technique is outlined which promises to form the basis of an exact dynamical theory for all topological defects described by a Nambu-Goto action. It is based on a method well known in the condensed matter literature: the u theory of Ohta, Jasnow, and Kawasaki [6], and its descendents [7]. In this approach, the defects are replaced by a multicomponent scalar field (the “ u ” field) specially designed so that its zeros track the positions of the defects. Although the equations of motion are nonlinear, when combined with a Gaussian ansatz for the field probability distribution, they are susceptible to a mean-field theory treatment. One can then calculate analytically many important quantities, such as the defect density and correlation functions, purely from the two-point correlation function of the u field, which is itself calculable.

To apply this theory to cosmic defects, all that is required is to make it relativistically covariant. The fictitious field that replaces the defects still has a Gaussian ansatz for its probability distribution function, and a self-consistent and self-similar solution for the linearized equations of motion can be found. With this in hand, one can calculate the defect density, using a generalization of well-known techniques, although the presence of time derivatives of the u field correlators complicates the procedure somewhat. In principle one could then go on to calculate more or less anything of interest: for example, in the cosmological context we would like to know the two-time correlation functions of the cosmic string stress-energy tensor [8]. However, in this Letter we shall content ourselves with calculating the scaling value

for the area density of domain walls, in an approximation where one neglects the time derivatives of the u -field correlation function. The results for walls are encouraging when compared to the numerical simulations [9–11]. The results for strings are more difficult to obtain, and will be presented elsewhere [12].

Another feature of the theory is that it describes the behavior of defects formed from initial conditions with a slight bias favoring one vacuum over another [10,11]. It is found that the defects disappear at a conformal time $\eta_c = \eta_i (U^2 / \langle u^2 \rangle_c)^{1/D}$ where $U = \langle u \rangle$ is the initial bias in the field, and $\langle u^2 \rangle_c$ the initial fluctuations around that value. Indeed, part of the motivation for this work was to account for this behavior observed in some interesting simulations recently carried out by Larsson, Sarkar, and White [11].

The essence of the technique is to replace the walls with a real scalar field, which vanishes precisely at the coordinates of the walls $X^\mu(\sigma^\alpha)$, where μ takes the values $0, \dots, D$ and α the values $0, \dots, D - 1$. Thus we begin with the equation $u(X^\mu) = 0$. Differentiating once with respect to the world-volume coordinates σ^α , we find

$$\partial_\beta X^\mu \partial_\mu u(X) = 0. \quad (1)$$

Thus the vector $\partial_\mu u(X)$ is spacelike normal to the wall.

The embedding of the p brane in the background space-time induces a metric in its world volume, $\gamma_{\alpha\beta} = g_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu$, where $g_{\mu\nu}$ is the space-time metric. Using the embedding metric we can covariantly differentiate (1) by acting with the operator $(-\gamma)^{-1/2} \partial_\alpha (-\gamma)^{1/2} \gamma^{\alpha\beta}$ to obtain

$$\square X^\mu \partial_\mu u + \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \partial_\mu \partial_\nu u = 0, \quad (2)$$

where \square is the covariant d’Alembertian and $\gamma = \det \gamma_{\alpha\beta}$. The defect equations of motion are [1,2]

$$\square X^\mu + \Gamma_{\nu\rho}^\mu \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu = 0, \quad (3)$$

where $\Gamma_{\nu\rho}^\mu$ is the affine connection. In Eq. (3) we can identify the tensor $P_{\mu\nu}^\parallel = \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu$, which is the tangential projector onto the wall. We can replace this

by $g^{\mu\nu} - P_{\mu\nu}^\perp$, where $P_{\mu\nu}^\perp = \partial_\mu u \partial_\nu u / (\partial u)^2$. Using the defect equations of motion (3) we may then write

$$[(\partial u)^2 g^{\mu\nu} - \partial^\mu u \partial^\nu u](\partial_\mu \partial_\nu u - \Gamma_{\mu\nu}^\rho \partial_\rho u) = 0. \quad (4)$$

This is the fundamental equation of motion for the field u which replaces the defects. It is actually unnecessary that this field has anything to do with the underlying Higgs fields, and so it may be called a ‘‘fictitious’’ field. In other approaches, which apply to global defects, the Higgs field ϕ is related to the fictitious field by the nonlinear transformation $\phi(\mathbf{x}, t) = f(u)$, where f solves the static defect field equations as a function of the transverse coordinate u [7].

The equations of motion (4) are not easy to solve, as they are nonlinear. However, they have the distinct advantage over the original equations of motion (3) in that they are local. If one tries to solve (3) directly, one must include the nonlocal processes of self-intersection and reconnection. In numerical simulations this is very time consuming, and involves the introduction of a seemingly arbitrary parameter p , which is the probability of reconnection. For strings, this is usually set to 1, following simulations of the full classical field theory [13]. In the current approach, reconnection happens with probability 1, provided the u -field configuration is smooth. This is an essentially geometric result. Where two walls intersect, the spatial component of the normal to the surface $u = 0$ changes direction discontinuously. If the second derivative of u exists, $\partial_i u$ must therefore vanish along the line of intersection, with no generic requirement on $\partial_0 u$. Hence as the field evolves u changes sign at the intersection point, which means that the walls generically reconnect or intercommute (see Fig. 1).

In order to solve the nonlinear field equations for u , the approach taken in the condensed matter context is essentially that of mean-field theory: bilinears in the field u are replaced by their averages. Thus one is assuming that the field is a Gaussian random field, and remains so throughout its evolution. This seems a good starting point for the relativistic version as well, although it should be borne in mind that the approximation is not well controlled.

We begin the mean-field theory manipulations by defining the basic equal-time two-point correlation function:

$$\langle u(\mathbf{x}, \eta) u(\mathbf{x}', \eta) \rangle = C(|\mathbf{x} - \mathbf{x}'|, \eta), \quad (5)$$

where the angle brackets denote an average over a spatially isotropic Gaussian probability distribution function. We shall also define $M_{\mu\nu}$ to be the equal-time two-point

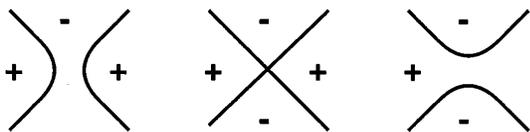


FIG. 1. The sign of the u field near the intersection of two walls. Generically, the sign of u changes at the intersection point as the intersection happens: hence the walls emerge with their connectivity changed.

correlator of $\partial_\mu u$:

$$\langle \partial_\mu u(\mathbf{x}, \eta) \partial_\nu u(\mathbf{x}', \eta) \rangle = M_{\mu\nu}(|\mathbf{x} - \mathbf{x}'|, \eta). \quad (6)$$

A two-point correlator with three derivatives will also be useful:

$$\langle \partial_\mu u(\mathbf{x}, \eta) \partial_\nu \partial_\rho u(\mathbf{x}', \eta) \rangle = \gamma_{\mu\nu\rho}(|\mathbf{x} - \mathbf{x}'|, \eta). \quad (7)$$

When referred to with no explicit spatial variables, the correlators are to be taken at zero separation. In this coincident limit, the assumed spatial isotropy of the distribution function dictates their forms. The nonzero components of $M_{\mu\nu}$ are

$$M_{00} = T(\eta), \quad M_{mn} = S(\eta) \delta_{mn}, \quad (8)$$

where $m, n = 1, \dots, D$, and the nonzero components of $\gamma_{\mu\nu\rho}$ are

$$\begin{aligned} \gamma_{000}(\eta) &= \frac{1}{2} \dot{T}(\eta), & \gamma_{0mn}(\eta) &= -\frac{1}{2} \dot{S}(\eta) \delta_{mn}, \\ \gamma_{m0n}(\eta) &= \frac{1}{2} \dot{S}(\eta) \delta_{mn}. \end{aligned} \quad (9)$$

We now linearize the equations of motion by taking the Gaussian average, and then find a self-consistent solution for the fields $u(\mathbf{x}, \eta)$. We need the following identities:

$$\langle (\partial u)^2 \partial_\mu \partial_\nu u \rangle = M \partial_\mu \partial_\nu u + 2\gamma_{\rho\mu\nu} g^{\rho\sigma} \partial_\sigma u, \quad (10)$$

$$\langle (\partial u)^2 \partial_\rho u \rangle = M \partial_\rho u + 2g^{\rho\sigma} M_{\eta\rho} \partial_\sigma u, \quad (11)$$

where $M = M_{\mu\nu} g^{\mu\nu}$.

In a flat Friedmann-Robertson-Walker space-time, the affine connection is $\Gamma_{\mu\nu}^\rho = (\delta_\mu^\rho \delta_\nu^0 + \delta_\nu^\rho \delta_\mu^0 - g_{\mu\nu} g^{\rho 0})(\dot{a}/a)$. We can now see that the linearized equations of motion have the form

$$\ddot{u} + \frac{\mu(\eta)}{\eta} \dot{u} - v^2 \nabla^2 u = 0, \quad (12)$$

where $\mu(\eta)$ and v depend on D . For Friedmann models, one can show that

$$\mu(\eta) = -2\eta(\dot{S}/S) + \alpha(\eta)[D - 3(T/S)], \quad (13)$$

$$v^2 = [D - 1 - (T/S)]/D, \quad (14)$$

where $\alpha(\eta) = \eta \dot{a}/a$.

In a scaling solution we would expect S and T to have a power-law behavior with η . Thus, so long as we are not near a transition in the equation of state of the Universe, μ and v^2 are constant. Imposing the boundary condition that u be regular as $\eta \rightarrow 0$, (12) then has the simple solution

$$u_{\mathbf{k}}(\eta) = A_\nu \left(\frac{\eta}{\eta_i} \right)^{(1-\mu)/2+\nu} \frac{J_\nu(kv\eta)}{(kv\eta)^\nu}, \quad (15)$$

where $A_\nu = 2^\nu \Gamma(\nu + 1) u_{\mathbf{k}}(\eta_i)$, and $(1 - \mu)^2/4 = \nu^2$. The form of the initial power spectrum $P_i(k) = |u_{\mathbf{k}}(\eta_i)|^2$ is taken to be white noise, which ensures that the field is spatially uncorrelated to begin with.

We may now evaluate T/S and v^2 , and implicitly solve for ν . First, we must decide the sign of ν . It turns out that it is inconsistent to take $(1 - \mu)/2 = \nu$ [one way of showing this is to compare $\dot{C}(\eta)$ calculated by explicit differentiation, and by evaluation of $\langle u\dot{u} \rangle$]. Thus C scales as η^{-D} , S and T as $\eta^{-(D+2)}$. Using standard integrals of Bessel functions, and defining the parameter $\beta = 2\nu - D - 1$, we find

$$\frac{T}{S} = \frac{(D+2)(D-1)}{2(D+2+\beta)}, \quad (16)$$

provided $\beta > 0$, so that the integrals for S and T are defined. To find β , we solve the equation

$$\mu \equiv \beta + D + 2 = \alpha D [1 - 3(T/S)] + 2(D+2). \quad (17)$$

The radiation-dominated ($\alpha = 1$) and matter-dominated ($\alpha = 2$) Friedmann models require a certain amount of algebra; however, Minkowski space ($\alpha = 0$) has the simple solution $\beta = D + 2$, giving $T/S = (D-1)/4$ and $v^2 = 3(D-1)/4D$.

Armed with the ‘‘mean-field’’ solution for $u(\mathbf{x}, \eta)$ we can now calculate anything that can be expressed in terms of local functions of the field and its derivatives, provided of course that we are able to perform the Gaussian integrals involved. In this Letter we content ourselves merely with evaluating the area density, which is given by

$$\mathcal{A} = \int d^D \sigma \sqrt{-\gamma} \delta_{D+1}(x - X(\sigma)). \quad (18)$$

Making the coordinate transformation from x^μ to (σ^α, u) near the wall, this can be rewritten as

$$\mathcal{A} = \delta(u) |\partial u|. \quad (19)$$

In order to calculate the Gaussian average of \mathcal{A} , one takes Fourier transforms [6]:

$$\langle \mathcal{A} \rangle = - \frac{\Gamma(D/2)}{2\pi^{1+D/2}} \int \frac{dk}{2\pi} \int \frac{d^{D+1}q}{(q^2)^{D/2}} \times \left\langle \frac{\partial}{\partial q} \cdot \frac{\partial}{\partial q} e^{iq \cdot \partial u + iku} \right\rangle. \quad (20)$$

One potential difficulty is that q^2 can vanish, due to the Lorentzian metric. The equation must therefore formally be defined by analytic continuation to imaginary time, and an accompanying Wick rotation of the time components of the Fourier transform variable q^μ .

One can show that the correlators involving time derivatives always appear in the combination $T/S - \dot{C}^2/CS$, which has the value $\frac{1}{4}$ in Minkowski space [13]. Neglecting these terms turns out to make only a 10% difference to the result for the averaged comoving defect area density $\mathcal{A}(\eta)$, which is

$$\langle \mathcal{A}(\eta) \rangle = \sqrt{2} \frac{\Gamma[(D+1)/2]}{\Gamma(D/2)} \left(\frac{S}{2\pi C} \right)^{1/2} + O(T/S - \dot{C}^2/CS). \quad (21)$$

In the limit $T/S - \dot{C}^2/CS \rightarrow 0$ we recover the original condensed matter result for domain walls [6,14,15],

as expected. The full calculation will be detailed elsewhere [12].

One can show that

$$\frac{S}{C} = \frac{1}{\eta^2} \frac{(\beta+1)(\beta+1+D/2)}{2\beta v^2}, \quad (22)$$

and we immediately see that the correct scaling behavior for the wall area density [$\mathcal{A}(\eta) \propto \eta^{-1}$] is reproduced. Furthermore, we are able to compute the coefficient, and compare with numerical simulations. The comoving area densities for walls in two- and three-dimensional Minkowski space, radiation- and matter-dominated Friedmann models are displayed in Table I.

The theory seems to work well: scalar field theory simulations in $D = 3$ give a comoving area density of approximately $1.5/\eta$ in both radiation [10,11] and matter [9] eras, which sits nicely with the calculated values. However, the agreement is partly fortuitous, as there are probably significant errors in both the theoretical and numerical values. In $D = 2$ the agreement is also good, although there is evidence for a small deviation from the η^{-1} behavior in $D = 2$ [9,10].

One may also ask how the network behaves when a small bias is introduced into the initial conditions, that is, if $\langle u(\mathbf{x}, \eta_i) \rangle = U$. In numerical simulations of domain walls [10,11], it is found that even for very small initial biases, for which the walls percolate, the system still evolves away from the percolating state and eventually the large walls break up and disappear. Similar behavior is well known in the study of quenches of condensed matter systems with a nonconserved order parameter [6,14,16,17].

The theoretical description of this behavior is fairly straightforward. Introducing a bias into the initial conditions alters the Gaussian average of (20) to $\langle \mathcal{A}(\eta) \rangle_U = \langle \mathcal{A}(\eta) \rangle \exp[-U^2/2C(\eta)]$, where $\langle \mathcal{A}(\eta) \rangle$ is the zero bias result from (21). If the system is close to being self-similar at some initial time η_i when the magnitude of the bias is U and the fluctuation around that value is $C(\eta_i)$, one can calculate the time η_c at which the defect density falls to a fraction e^{-1} of its scaling value to be

$$\eta_c = \eta_i [U^2/2C(\eta_i)]^{-1/D}. \quad (23)$$

Recent simulations by Larsson, Sarkar, and White lend support to the above calculations of $\langle \mathcal{A}(\eta) \rangle_U$ and η_c [11]. Coulson *et al.* [10] did not attempt fits of the correct form, $\mathcal{A} \propto \eta^{-1} \exp[-(\eta/\eta_c)^D]$, to their simulations,

TABLE I. The calculated values of the comoving domain wall area density in the self-consistent linearized solution for u . Values listed are for Minkowski space ($\alpha = 0$), and radiation and matter-dominated FRW models ($\alpha = 1, 2$, respectively).

α	$\mathcal{A} (D = 2)$	$\mathcal{A} (D = 3)$
0	$1.6\eta^{-1}$	$1.9\eta^{-1}$
1	$1.7\eta^{-1}$	$2.1\eta^{-1}$
2	$1.9\eta^{-1}$	$2.3\eta^{-1}$

although they did note that the walls disappeared faster than a simple exponential in $D = 3$.

To summarize, this Letter outlines a new analytic technique for describing the dynamics of topological defects after a cosmological phase transition. It is a relativistic version of a well-known approach in condensed matter physics, due to Ohta, Jasnów, and Kawasaki [6]. The scaling area density of domain walls in three spatial dimensions is calculated, and agrees quantitatively with numerical simulations of a real scalar field [9–11]. Further work is clearly needed, particularly on strings, but it already seems clear that developments of this technique will be of great benefit to the study of cosmic defects.

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- [1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge Univ. Press, Cambridge, 1994).

- [2] M. Hindmarsh and T.W.B. Kibble, *Rep. Prog. Phys.* **55**, 478 (1995).
- [3] E.W. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, 1990).
- [4] N. Turok and D.N. Spergel, *Phys. Rev. Lett.* **66**, 3093 (1991).
- [5] J.A.N. Filipe and A.J. Bray, *Phys. Rev. E* **50**, 2523 (1994).
- [6] T. Ohta, D. Jasnów, and K. Kawasaki, *Phys. Rev. Lett.* **49**, 1223 (1982).
- [7] A. Bray, in *Formation and Interactions of Topological Defects*, NATO ASI Series, edited by R. Brandenberger and A.-C. Davis, (Plenum, New York, 1995); N. Goldenfeld, *ibid.*; G. Mazenko, *ibid.*
- [8] S. Veeraraghavan and A. Stebbins, *Astrophys. J.* **365**, 37 (1990).
- [9] W.H. Press, B.S. Ryden, and D.N. Spergel, *Astrophys. J.* **347**, 590 (1989).
- [10] D. Coulson, Z. Lalak, and B. Ovrut, *Phys. Rev. D* **53**, 4237 (1996).
- [11] S. Larsson, S. Sarkar, and P.L. White, Oxford Report No. OUTF-96-11P, 1996.
- [12] M. Hindmarsh and E. Copeland (unpublished).
- [13] E.P.S. Shellard, *Nucl. Phys.* **B283**, 264 (1988); R. Matzner and J. McCracken, in *Cosmic Strings: the Current Status* (World Scientific, Singapore, 1988), pp. 32–41.
- [14] H. Toyoki and K. Honda, *Phys. Rev. B* **33**, 385 (1986).
- [15] F. Liu and G.F. Mazenko, *Phys. Rev. B* **46**, 5963 (1992).
- [16] M. Mondello and N. Goldenfeld, *Phys. Rev. A* **45**, 657 (1992).
- [17] J.A.N. Filipe, A.J. Bray, and S. Puri, *Phys. Rev. E* **52**, 6082 (1995).